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ON THE VANISHING DISPLACEMENT CURRENT LIMIT FOR TIME-HARMONIC MAXWELL EQUATIONS

Peter Quell, Martin Reißel

Bericht 96 - 159 März 1996

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Peter Quell, Martin Reißel
Fachbereich Mathematik
Universi tät Kaiserslautern
D-67653 Kaiserslautern
Germany

Abstract

This paper consider a transmission boundary-value problem for the time-harmonic Maxwell equations neglecting displacement currents which is frequently used for the numerical computation of eddy-currents. Across material boundaries the tangential components of the magnetic field H and the normal component of the magnetization μH are assumed to be continuous. This problem admits a hyperplanc of solutions if the domains under consideration are multiply connected. Using integral equation methods and singular perturbation theory it is shown, that this hyperplane contains a unique point which is the limit of the classical electromagnetic transmission boundary-value problem for vanishing displacement currents. Considering the convergence proof, a simple constructive criterion how to select this solution is immediately derived.

Key words. time-harmonic Maxwell equations, transmission boundary-value problems. tnultiply connected domains, asymptotic analysis, singular perturbations, integral equation methods

AMS subject classification. 35Q60, 45F99, 78A25

1 Introduction

The investigations presented here were initiated by a common project with a. company manufacturing generators for power plants. During the design phase of these machines, the minimization of the losses caused by induced eddy-currents is one of the primary goals. Since prototyping and measurements are extremely expensive, the use of numerical simulation software is gaining more and more importance.

Mathematically, the situation can be described by the setting displayed below.

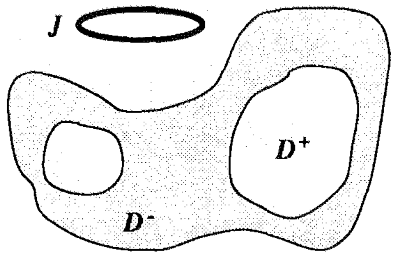

Figure 1, Eddy-current problem

A bounded domain of conducting material $D^- \subset \mathbb{R}^3$ is surrounded by some insulator in $D^+ = \mathbb{R}^3 \setminus \bar{D}^-$, D^+ connected. In D^+ a current of density J is given (usually a coil). J produces an electromagnetic field which vice versa induces eddy-currents in D^-.

Since generators usually rotate at a constant frequency, a classical electrodynamical treatment of the above setting delivers the following transmission boundary-value problem for the time-harmonic Maxwell equations.

$$\begin{array}{ll} \operatorname{curl} H^+ = J - i\omega\varepsilon^+ E^+ & \\ \operatorname{curl} E^+ = i\omega\mu^+ H^+ & \text{in } D^+ \end{array} \qquad \begin{array}{ll} \operatorname{curl} H^- = (\sigma^- - i\omega\varepsilon^-)E^- & \\ \operatorname{curl} E^- = i\omega\mu^- H^- & \text{in } D^-, \end{array}$$

$$\begin{array}{l} n \wedge H^+ = n \wedge H^- \\ n \wedge E^+ = n \wedge E^- \end{array} \quad \text{on } \Gamma = \partial D^\pm, \tag{1}$$

$$\omega\mu^+ H^+(x) \wedge \frac{x}{|x|} - k^+ E^+(x) = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty,$$

where n is the outer unit normal to the boundary of D^-, w is the frequency $\mu^\pm, \varepsilon^\pm$ are the usual material parameters and $k^+ = \sqrt{\omega^2\mu^+\varepsilon^+}$ the wavenumber in D^+. It is well known, that under certain restrictions on J, the material behaviour and the domains $D^\pm$, (1) possesses a unique solution [7].

Considering the typical size of the frequency and the material constants of (1)

$$w \approx 2\pi \cdot 50\text{Hz}, \qquad \varepsilon^\pm \approx 10^{-11}\frac{\text{As}}{\text{Vm}}, \qquad \mu^\pm \approx 10^{-6}\frac{\text{Vs}}{\text{Am}}, \qquad \sigma^- \approx 10^7\frac{\text{V}}{\text{Am}}$$

we see, that $\varepsilon^\pm$ are small parameters which introduce some kind of stiffness in the numerical treatment of the problem. Thus, neglecting the displacement currents $-i\omega\varepsilon^\pm E^\pm$ the corresponding asymptotic equations are considered. Since the equations in D^+ change to Pre-Maxwell type, also the radiation conditions are modified to $H^+(x), E^+(x) = o(1)$ for $|x| \to \infty$.

Moreover, it is common to replace the boundary condition $n \wedge E^+ = n \wedge E^-$ by $n \cdot (\mu^+ H^+) = n \cdot (\mu^- H^-)$ on Γ. Together we get the system of equations

$$\begin{array}{ll} \operatorname{curl} H^+ = J & \\ \operatorname{curl} E^+ = i\omega\mu^+ H^+ & \text{in } D^+ \end{array} \qquad \begin{array}{ll} \operatorname{curl} H^- = \sigma^- E^- & \\ \operatorname{curl} E^- = i\omega\mu^- H^- & \text{in } D^-, \end{array}$$

$$\begin{array}{l} n \wedge H^+ = n \wedge H^- \\ n \cdot (\mu^+ H^+) = n \cdot (\mu^- H^-) \end{array} \quad \text{on } \Gamma, \tag{2}$$

$$H^+(x) = o(1), \qquad E^+(x) = o(1), \qquad |x| \to \infty,$$

which is frequently found in the engineering literature [3] as a model for eddy-current problems.

For the numerical treatment (2) is reformulated by eliminating J with the help of the law of Biot-Savart. The resulting set of equations in the unknowns $H^\pm$ is

$$\begin{matrix} \operatorname{curl} H^+ = 0 \\ \operatorname{div} H^+ = 0 \end{matrix} \quad \text{in } D^+ \qquad \operatorname{curl}\left(\frac{1}{\sigma^-}\operatorname{curl} H^-\right) = i\omega\mu^- H^- \quad \text{in } D^-,$$

$$\begin{matrix} n \wedge H^+ - n \wedge H^- = c \\ n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) = f \end{matrix} \quad \text{on } \Gamma, \tag{3}$$

$$H^+(x) = o(1), \qquad |x| \to \infty,$$

where c, f essentially depend on J. To handle the differential equations in D^+, the following argument was used.

$$\operatorname{curl}\ H^+ = 0 \Rightarrow H^+ = \nabla\varphi \qquad \text{and} \qquad \operatorname{div}\ H^+ = 0 \Rightarrow \Delta\varphi = 0.$$

But this is only valid if $D^\pm$ are simply connected. For topological genus $p > 0$ the magnetic field is $H^+ = \nabla\varphi + \sum_{i=1}^{p} h_i Z_+^i$, where Z_+^i are the p Neumann fields of D^+ (for the definition of Z_+^i see Lemma 1 below) and $h_i \in \mathbb{C}$. As was shown by one of the authors in [8], the h_i are free parameters in the sense that they are not determined neither by (2) nor by (3).

Following the above strategy (ail h_i are 0) does in general not lead to satisfactory results if $D^\pm$ arc multiply connected. Consider for example the massive aluminium cube of figure 2 which is surrounded by a concentric square loop.

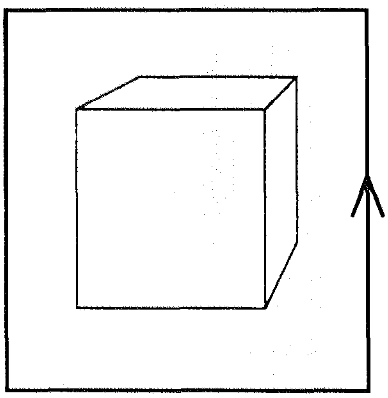

Figure 2., Massive cube

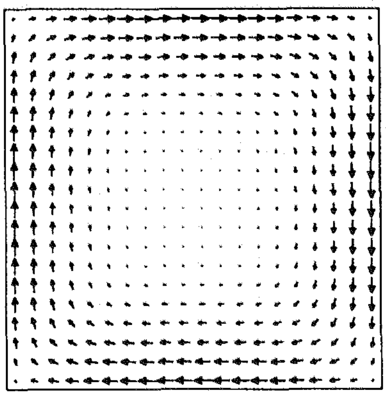

Figure 3., Computed eddy-currents

$D^\pm$ are simply connected and the computed eddy-currents in the plane of the loop show the behaviour which is expected by classical electrodynamics (figure 3).

Now we cut a small hole into the cube (figure 4) so that, the topological genus changes from 0 to 1.. Let us take the value 0 for the free parameter (which is actually done automatically if you use the above reasoning). Due to continuity properties, the true eddy-currents in the plane of the loop should not change too much. But in fact, the numerical results show a very strange behaviour (figure 5).

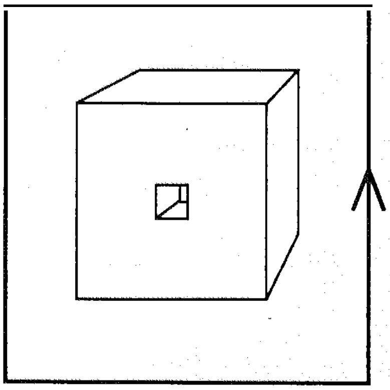

Figure 4., Modified cube

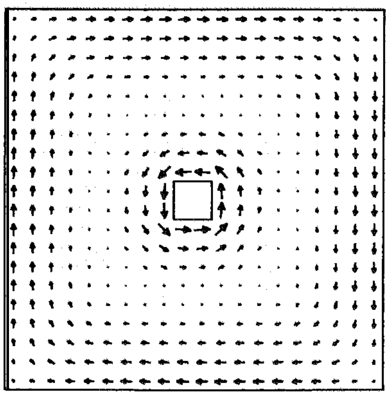

Figure 5., Computed eddy-currents

So the central question now is, how to determine the free parameters correctly.

In this paper, we show by using integral equation methods, that for $\partial D^{\pm}$ being C^2 and linear, homogeneous, isotropic materials in $D^{\pm}$ the solution of (1) converges for $\varepsilon \to 0$ in a certain Holder norm to a specific solution of (2) resp. (3). This limiting procedure uniquely determines the free parameters of (2) resp. (3). From the proofs, a numerical algorithm for the computation of these parameters can be extracted. It is easily added to existing software, thus giving the correct eddy-current distributions for arbitrary topological genus p (compare figure 5 with figure 6 for p = 1).

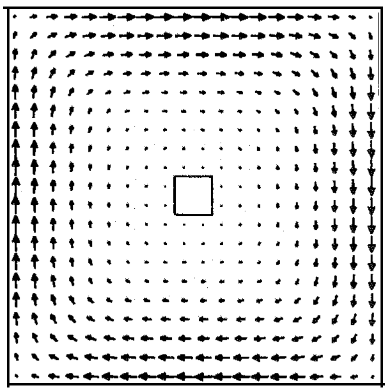

Figure 6., Corrected eddy-currents

2 Assumptions and Notations

In this section we are going to specify all the technical prerequisites and notations we will use throughout the rest of the paper.

We consider an open, bounded domain $D^- \subset \mathbb{R}^3$ consisting of m connected components D_j^-, $j \in \{1, \ldots, m\}$ of topological genus p_j with boundaries Γ_j of class C^2, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. The open complement $D^+ = \mathbb{R}^3 \backslash \bar{D}^-$ is assumed to be connected. The resulting topological genus of $D^{\pm}$ is $p = \sum_{j=1}^m p_j$ and $\Gamma = \partial D^{\pm} = \bigcup_{j=1}^m \Gamma_j$. There exist p surfaces $\Sigma_i^{\pm} \subset D^{\pm}$, $i \in \{1, \ldots, p\}$ so that $D^{\pm} \setminus \bigcup_{i=1}^p \Sigma_i^{\pm}$ are simply connected. The boundary curves $\gamma_i^{\pm} = \partial\Sigma_i^{\mp}$ are closed curves on Γ. Moreover D^J denotes an open bounded domain with ∂D^J being C^2 and $\bar{D}^J \subset D^+$.

All the material coefficients occuring in the Maxwell equations are assumed to be real, positive constants in the corresponding domains D^+ resp. D^-, unless stated otherwise.

Radiation conditions are always understood to hold uniformly for all directions $\frac{x}{|x|}$. Concerning integrals, we suppress the arguments of the integrands wherever there is no danger of confusion. For constants occuring in estimates we will frequently use the same name in each step, even if the value changes. Moreover, $\wedge$ is the standard vector product in $\mathbb{C}^3$, the complex conjugate of some number $z \in \mathbb{C}$ is denoted by $\bar{z}$ and we write $u \cdot v$ for $u^T v$, $u, v \in \mathbb{C}^3$.

For $\alpha \in (0, 1]$, $C^{0\alpha}(D)$ equipped with

$$\|f\|_{0\alpha,D} = \sup_{x\in D} |f(x)| + \sup_{x,y\in D, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}}$$

denotes the Banach space of bounded, uniformly Hölder continuous functions on D. Depending on the topology of $D^{\pm}$, there exist two classes of special harmonic vector fields [6].

Lemma 1 (Neumann fields) *Let p be the topological genus of $D^{\pm}$. There exist exactly p linearly independent vector fields $Z_i^{\pm} \in C^{\infty}(D^{\pm}) \cap C^{0\alpha}(\bar{D}^{\pm})$, $i \in \{1, \ldots, p\}$ with*

$$\operatorname{curl} Z_i^{\pm} = 0, \quad \operatorname{div} Z_i^{\pm} = 0 \quad \text{in } D^{\pm},$$
$$n \cdot Z_i^{\pm} = 0 \quad \text{on } \Gamma,$$
$$Z_i^{+}(x) = O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty,$$
$$\int_{\gamma_j^{\pm}} \tau \cdot Z_i^{\pm} dl = \delta_{ij}, \qquad \int_{\gamma_j^{\pm}} \tau \cdot Z_i^{\mp} dl = 0, \qquad \forall i, j \in \{1, \ldots, p\},$$

n being the outer unit normal to D^- on Γ.

Lemma 2 (Dirichlet fields) D^- consists of m connected components D_j^-. There exist exactly m linearly independent vector fields $Y_j^+ \in C^{\infty}(D^+) \cap C^{0\alpha}(\bar{D}^+)$, $j \in \{1, \ldots, m\}$ with

$$\operatorname{curl} Y_j^+ = 0, \quad \operatorname{div} Y_j^+ = 0 \quad \text{in } D^+,$$
$$n \wedge Y_j^+ = 0 \quad \text{on } \Gamma,$$
$$Y_j^+(x) = O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty.$$

They are given by $Y_j^+ = \operatorname{grad} \varphi_j$, $j \in \{1, \ldots, m\}$,

$$\Delta\varphi_j = 0 \qquad \text{in } D^+,$$
$$\varphi_j|_{\Gamma_i} = \delta_{ij}, \qquad i, j \in \{1, \ldots, m\},$$
$$\varphi_j(x) = o(1), \quad |x| \to \infty.$$

In the sequel, we will frequently use the following Banach spaces.

Definition 1

- $C_*^{0\alpha}(\Gamma) = \left\{ f \in C^{0\alpha}(\Gamma) | \int_{\Gamma_j} f ds = 0, \ \forall j \in \{1, \ldots, m\} \right\}$, *equipped with the norm* $\| . \|_{0\alpha,\Gamma}$
- $T^{0\alpha}(\Gamma) = \left\{ a = (a_1, a_2, a_3)^T | a \in C^{0\alpha}(\Gamma),\ n \cdot a = 0 \right\}$, $\|a\|_{T\alpha,\Gamma} = \|a\|_{0\alpha,\Gamma}$, *where* n *is the outer normal to* D^- *on* Γ
- $T_d^{0\alpha}(\Gamma) = \{ a \in T^{0\alpha}(\Gamma) | Div\, a \in C^{0\alpha}(\Gamma) \}$, *equipped with the norm* $\|a\|_{d\alpha,\Gamma} = \max\{ \|a\|_{T\alpha,\Gamma}, \|Div\, a\|_{0\alpha,\Gamma} \}$, *where* Div *denotes the surface divergence on* Γ
- $T_*^{0\alpha}(\Gamma) = \left\{ a \in T_d^{0\alpha}(\Gamma) | Div\, a = 0, \ \int_\Gamma a \cdot Z_i^+ ds = 0 \ \ \forall i \in \{1, \ldots, p\} \right\}$, *with the norm* $\| \cdot \|_{d\alpha\Gamma}$

Later on, we consider the dual systems

$$(C^{0\alpha}(\Gamma), C^{0\alpha}(\Gamma), < \cdot, \cdot >), \qquad < u, v >= \int_\Gamma uv\, ds \tag{4}$$

resp.

$$(T_d^{0\alpha}(\Gamma), T^{0\alpha}(\Gamma), < \cdot, \cdot >), \qquad < u, v >= \int_\Gamma u \cdot v\, ds. \tag{5}$$

Moreover, we introduce for $k \in \mathbb{C}$, $x, y \in \mathbb{R}^3$, $x \neq y$ the function

$$\Phi_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

so that $-\Phi_k(x, 0)$ is a fundamental solution of the Helmholtz operator $\triangle + k^2$ in $\mathbb{R}^3$ with wavenumber k. Throughout the whole paper we assume that $\mathrm{Im}(k) \geq 0$.

3 Sketch of the Paper

To establish the convergence proof mentioned in the introduction, we first consider a sequence of transmission boundary-value problems similar to (1),(2) where the current density J is removed at the expense of some additional inhomogeneities in the boundary conditions. All the results we are going to prove below are valid if $\varepsilon^+, \varepsilon^- > 0$ tend to 0 separately. For the sake of simplicity and to avoid cumbersome notations, we restrict ourselve to the case $\varepsilon^+ = \varepsilon^- = \varepsilon > 0$, $\varepsilon \to 0$. The system related to (1) is

Problem 1 *Let $\varepsilon > 0$. For given $c, d \in T_d^{0\alpha}(\Gamma)$ find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0\alpha}(\bar{D}^\pm$ so that*

$$\begin{array}{ll} curl\ H^+ = -i\omega\varepsilon E^+ & \\ curl\ E^+ = i\omega\mu^+ H^+ & in\ D^+ \end{array} \qquad \begin{array}{ll} curl\ H^- = (\sigma^- - i\omega\varepsilon)E^- & \\ curl\ E^- = i\omega\mu^- H^- & in\ D^-, \end{array}$$

$$\left.\begin{array}{l} n \wedge H^+ - n \wedge H^- = c \\ n \wedge E^+ - n \wedge E^- = d \end{array}\right. \quad on\ \Gamma,$$

$$\omega\mu^+ H^+(x) \wedge \tfrac{x}{|x|} - k^+ E^+(x) = o\left(\tfrac{1}{|x|}\right), \qquad |x| \to \infty,$$

where $k^+ = \sqrt{\omega^2\mu^+\varepsilon}$.

Problem 1 is well-posed [9]. We show, that the unique solution of Problem l converges in $C^{0\alpha}(\bar{D}^\pm)$ to tlıe unique solution of

Problem *2 For $c, d \in T_d^{0\alpha}(\Gamma)$ find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0\alpha}(\bar{D}^\pm)$ with*

$$\begin{array}{ll} curl\ H^+ = 0 & \\ curl\ E^+ = i\omega\mu^+ H^+ & in\ D+ \\ div\ E^+ = 0 & \end{array} \qquad \begin{array}{ll} curl\ H- = \sigma^- E^- & \\ curl\ E- = i\omega\mu^-\ H- & in\ D^-, \end{array}$$

$$\int_{\Gamma_j} n \cdot E^+ ds = 0 \qquad \forall j \in \{1, \ldots, m\},$$

$$\left.\begin{array}{l} n \wedge H^+ - n \wedge H^- = c \\ n \wedge E^+ - n \wedge E^- = d \end{array}\right. \quad on\ \Gamma,$$

$$H+(z) = o(l), \quad E^+(x) = o(l), \qquad |x| \to cm.$$

Moreover, every solution of the last problem also solves

Problem 3 *For- given $c \in T_d^{0\alpha}(\Gamma)$, $g \in C_*^{0\alpha}(\Gamma)$ and $h = (h_1, \ldots, h_p)^T \in \mathbb{C}^p$, find $H^\pm$, $E*$ $\in C^1(D^\pm) \cap C^{0\alpha}(\bar{D}^\pm)$ so that*

$$\begin{array}{ll} curl\ H^+ = 0 & \\ curl\ E^+ = i\omega\mu^+ H^+ & in\ D^+ \\ div\ E^+ = 0 & \end{array} \qquad \begin{array}{ll} curl\ H^- = \sigma^- E^- & \\ curl\ E^- = i\omega\mu^- H^- & in\ D^-, \end{array}$$

$$\int_{\Gamma_j} n \cdot E^+ ds = 0 \qquad \forall j \in \{1, \ldots, m\},$$

$$\int_{\gamma_i^+} \tau \cdot H^+ dl = h_i \qquad \forall i \in \{1, \ldots, p\},$$

$$\left.\begin{array}{l} n \wedge H^+ - n \wedge H^- = c \\ n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) = g \end{array}\right. \quad on\ \Gamma,$$

$$H^+(x) = o(1), \quad E^+(x) = o(1), \qquad |x| \to \infty$$

for suitable g and some special parameters h_i. Problem 3 corresponds to the set of equations (2) and the h_i represent the parameters we want to determine.

We first show in the next section, that Problem 3 is well posed. Using this result we also obtain unique solvability and continuous dependence on the data for Problem 2. In the remaining sections, we establish an integral operator equation of the type $\mathcal{L}_\varepsilon e_\varepsilon = f_\varepsilon$, $e_\varepsilon = n \wedge E_\varepsilon^+|_\Gamma$, E_ε^+ being the exterior electric field of the solution of Problem 1 for $\varepsilon > 0$ resp. Problem 2 and thus Problem 3 for $\varepsilon = 0$. Using the well-posedness of Problem 1 and Problem 2, we prove that $\mathcal{L}_\varepsilon, \varepsilon \geq 0$ has a bounded inverse. Moreover $\mathcal{L}_\varepsilon$ and f_ε converge in suitable norms to $\mathcal{L}_0$ and f_0, so that the solution e_ε of $\mathcal{L}_\varepsilon e_\varepsilon = f_\varepsilon$ converges to the solution e_0 of $\mathcal{L}_0 e_0 = f_0$ as $\varepsilon \to 0$. As a direct consequence, the tangential components on the boundary Γ of the solutions of Problem 1 converge to those of Problem 2 and thus to the tangential components of a special solution of Problem 3. Afterwards we show, that this is already enough to get convergence of the fields in $C^{0\alpha}(\bar{D}^\pm)$.

In the multiply connected case, the definition of $\mathcal{L}_0$ resp. the convergence of $\mathcal{L}_\varepsilon$ to $\mathcal{L}_0$ causes some additional difficulties which are due to a singular perturbation of one of the operators involved. To outline the basic ideas, we first treat the case of simply connected domains and present the much more technical part for multiply connected $D^\pm$ separately.

Finally we return to our original problems (1),(2) and use the results obtained so far to prove convergence of the solutions of

Problem 4 *Let $\varepsilon > 0$. For given $J \in C^1(\mathbb{R}^3)$, div $J = 0$, supp$(J) \subset D^J$ find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0\alpha}(\bar{D}^\pm)$ so that*

$$\begin{aligned} &\text{curl } H^+ = J - i\omega\varepsilon E^+ \\ &\text{curl } E^+ = i\omega\mu^+ H^+ \end{aligned} \quad \text{in } D^+ \qquad \begin{aligned} &\text{curl } H^- = (\sigma^- - i\omega\varepsilon)E^- \\ &\text{curl } E^- = i\omega\mu^- H^- \end{aligned} \quad \text{in } D^-,$$

$$\begin{aligned} n \wedge H^+ &= n \wedge H^- \\ n \wedge E^+ &= n \wedge E^- \end{aligned} \quad \text{on } \Gamma,$$

$$\omega\mu^+ H^+(x) \wedge \frac{x}{|x|} - k^+ E^+(x) = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty,$$

where $k^+ = \sqrt{\omega^2\mu^+\varepsilon}$

to a solution of

Problem 5 *For given $J \in C^1(\mathbb{R}^3)$, $div\ J = 0$, $supp(J) \subset D^J$ and $h = (h_1, \dots, h_p)^T \in \mathbb{C}^p$, find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0\alpha}(\bar{D}^\pm)$ so that*

$$\begin{aligned} curl\ H^+ &= J \\ curl\ E^+ &= i\omega\mu^+ H^+ \quad in\ D^+ \\ div\ E^+ &= 0 \end{aligned} \qquad \begin{aligned} curl\ H^- &= \sigma^- E^- \\ curl\ E^- &= i\omega\mu^- H^- \end{aligned} \quad in\ D^-,$$

$$\int_{\Gamma_j} n \cdot E^+ ds = 0 \qquad \forall j \in \{1, \dots, m\},$$

$$\int_{\gamma_i^+} \tau \cdot H^+ dl = h_i \qquad \forall i \in \{1, \dots, p\},$$

$$\begin{aligned} n \wedge H^+ &= n \wedge H^- \\ n \cdot (\mu^+ H^+) &= n \cdot (\mu^- H^-) \end{aligned} \quad on\ \Gamma,$$

$$H^+(x) = o(1), \ E^+(x) = o(1), \qquad |x| \to \infty$$

for a specific choice of the circulations h_i.

4 Well-Posedness of Problem 3

A transmission boundary-value problem similar to Problem 3 was already investigated by one of the authors in [8]. For the following chapters we need some detailed information on the continuous dependence on the given data, which are not included there. These are finally obtained by some modifications of the proofs contained in [S]. It turns out, that things are even simplified in comparison to [S] and that some of the newly obtained intermediate results can also be used in later chapters. To keep the presentation short, we refer to [S] wherever possible.

A first uniqueness result for Problem 3 is found in [S].

Lemma 3 *Two solutions of Problem 3 to the same data c, g, h coincide in the fields H^+, H^- and E^-.*

To prove existence and continuous dependence of the solution on the boundary data c and g, we first, solve the case $h_i = 0\ \forall i \in \{1, \dots, p\}$. We consider the following auxiliary problem, which is obtained from Problem 3 by discarding E^+.

Problem 6 *For $c \in T_d^{0\alpha}(\Gamma)$, $g \in C_*^{0\alpha}(\Gamma)$ given, find a solution $H^\pm \in C^1(D^\pm) \cap$*

$C^{0\alpha}(\bar{D}^{\pm})$, $E^- \in C^1(D^-) \cap C^{0\alpha}(\bar{D}^-)$ *of*

$$\begin{array}{llll} \text{curl } H^+ = 0 & \text{in } D^+ & \text{curl } H^- = \sigma^- E^- & \text{in } D^-, \\ \text{div } H^+ = 0 & & \text{curl } E^- = i\omega\mu^- H^- & \end{array}$$

$$\int_{\gamma_i^t} \tau \cdot H^+ dl = 0 \qquad \forall i \in \{1, \ldots, p\},$$

$$\left.\begin{array}{r} n \wedge H^+ - n \wedge H^- = c \\ n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) = g \end{array}\right\} \quad on\ \Gamma,$$

$$H^+(x) = o(1), \qquad |x| \to \infty.$$

The following Lemma was shown in [8].

Lemma 4 *Problem 6 has at most one solution,*

To reduce Problem 6 to boundary integral equations, we will use the following operators.

Definition 2 For $a \in T^{0\alpha}(\Gamma)$, $b \in T_d^{0\alpha}(\Gamma)$, $\lambda \in C^{0\alpha}(\Gamma)$, $k \in \mathbb{C}$, $Im(k) \geq 0$ *we define*

$$\begin{aligned}
(M_k a)(x) &= 2n(x) \wedge \int_\Gamma curl_x\, (a(y)\Phi_k(x,y))\, ds(y), \\
(M_k' a)(x) &= n(x) \wedge M_k(n \wedge a)(x), \\
(N_k a)(x) &= 2n(x) \cdot \int_\Gamma curl_x\, (a(y)\Phi_k(x,y))\, ds(y), \\
(U_k a)(x) &= 2n(x) \wedge \int_\Gamma a(y)\Phi_k(x,y) ds(y), \\
(K_k \lambda)(x) &= 2\int_\Gamma \lambda(y) \frac{\partial}{\partial n(y)} \Phi_k(x,y) ds(y), \\
(K_k' \lambda)(x) &= 2\int_\Gamma \lambda(y) \frac{\partial}{\partial n(x)} \Phi_k(x,y) ds(y), \\
(S_k \lambda)(x) &= 2\int_\Gamma \lambda(y) \Phi_k(x,y) ds(y), \\
(P_k \lambda)(x) &= 2n(x) \wedge \int_\Gamma n(y)\lambda(y)\Phi_k(x,y) ds(y), \\
(Q_k \lambda)(x) &= 2n(x) \cdot \int_\Gamma n(y)\lambda(y)\Phi_k(x,y) ds(y), \\
(R_k \lambda)(x) &= 2n(x) \wedge \int_\Gamma grad_x\, (\lambda(y)\Phi_k(x,y))\, ds(y), \\
(T_k b)(x) &= (R_k(Div\ b))\,(x) + k^2\,(U_k b)\,(x).
\end{aligned}$$

For $Im(k) > 0$ *we set* $D_k\lambda = k^2(I - K_k)^{-1} S_k$.

According to [1], we get the following mapping properties.

Lemma 5 *For the linear operators defined above holds*

$$\begin{array}{rll} M_k & : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma) & \text{compact,} \\ M_k' & : T^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma) & \text{compact,} \\ N_k & : T_d^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma) & \text{bounded,} \\ U_k, T_k & : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma) & \text{bounded,} \\ K_k, K_k', S_k, Q_k, D_k & : C^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma) & \text{compact,} \\ P_k, R_k & : C^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma) & \text{bounded,} \\ R_k - R_0 & : C^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma) & \text{compact.} \end{array}$$

M_k, M_k' are adjoint with respect to the dual system (5) and K_k, K_k' are adjoint with respect to (4). Moreover

$$< T_k a, n \wedge b > = < n \wedge a, T_k b > \qquad \forall a, b \in T_d^{0\alpha}(\Gamma),$$

where $< \cdot, \cdot >$ is the bilinear form of (5).

Now let us consider the vector fields

$$\begin{aligned} H^+(x) &= \int_\Gamma \operatorname{grad}_x (\lambda(y)\Phi_0(x,y))\, ds(y), \\ H^-(x) &= \int_\Gamma \operatorname{grad}_x (\lambda(y)\Phi_k(x,y))\, ds(y) + \int_\Gamma n(y)(D_k\lambda)(y)\Phi_k(x,y) ds(y) \\ &\quad + \int_\Gamma \operatorname{curl}_x (a(y)\Phi_k(x,y))\, ds(y), \\ E^-(x) &= \frac{1}{\sigma^-}\left(\operatorname{curl} H^-\right)(x), \end{aligned} \tag{6}$$

where $k = \sqrt{i\omega\mu^-\sigma^-}$, $\operatorname{Im}(k) > 0$.

Lemma 6 *If $a \in T_d^{0\alpha}(\Gamma)$, $\lambda \in C_*^{0\alpha}(\Gamma)$ the fields (6) fulfill the following conditions.*

(i) $H^\pm \in C^2(D^\pm) \cap C^{0\alpha}(\bar{D}^\pm)$.

(ii) $\operatorname{curl} H^+ = 0$, $\quad \operatorname{div} H^+ = 0 \quad$ in D^+, $H^+(x) = o(1)$ and

$$\int_{\gamma_i^+} \tau \cdot H^+ dl = 0 \qquad \forall i \in \{1, \ldots, p\}.$$

(iii) $\operatorname{curl} H^-$ can be extended to $C^{0\alpha}(\bar{D}^-)$ and $\operatorname{div} H^- = 0$ in D^-.

(iv) $\operatorname{curl} E^- = i\omega\mu^- H^-$.

(v) There exists a constant c_α independent of a, λ, δ, so that

$$\begin{aligned} \|H^-\|_{0\alpha,\bar{D}^-} &\le c_\alpha \max\{\|a\|_{d\alpha,\Gamma}, \|\lambda\|_{0\alpha,\Gamma}\}, \\ \|\operatorname{div} H^-\|_{0\alpha,\bar{D}^-} &\le c_\alpha \|\lambda\|_{0\alpha,\Gamma}, \\ \|\operatorname{curl} H^-\|_{0\alpha,\bar{D}^-} &\le c_\alpha \max\{\|a\|_{d\alpha,\Gamma}, \|\lambda\|_{0\alpha,\Gamma}\}, \\ \|H^+\|_{0\alpha,\bar{D}^+} &\le c_\alpha \|\lambda\|_{0\alpha,\Gamma}. \end{aligned}$$

Proof: (i), (ii), the first part of (iii), (iv) and (v) follow directly from the properties of the layer potentials shown in [1]. It remains to be proven, that div H^- vanishes in D^-. For $x \in D^-$ we obtain

$$\begin{aligned}\left(\operatorname{div} H^-\right)(x) &= \Delta_x \int_\Gamma \lambda(y)\Phi_k(x,y)ds(y) + \int_\Gamma \operatorname{div}_x \left(n(y)(D_k\lambda)(y)\Phi_k(x,y)\right) ds(y) \\ &= -k^2 \int_\Gamma \lambda(y)\Phi_k(x,y)ds(y) + \int_\Gamma (D_k\lambda)(y)\, n(y) \cdot \operatorname{grad}_x \Phi_k(x,y)ds(y).\end{aligned}$$

Using the jump relations for the layer potentials [1], we get

$$\left(\operatorname{div} H^-\right)(x)\,|_\Gamma = -\frac{1}{2}k^2(S_k\lambda)(x) + \frac{1}{2}\left((I - K_k)D_k\lambda\right)(x) = 0$$

since $D_k = k^2(I - K_k)^{-1}S_k$. Moreover, div H^- solves the scalar Helmholtz equation in D^- with wavenumber k, $\operatorname{Im}(k) > 0$, so that by the standard uniqueness result for this case the proof is completed. ∎

Thus we observe, that for $a \in T_d^{0\alpha}(\Gamma)$, $\lambda \in C_*^{0\alpha}(\Gamma)$, $H^\pm$, E^- defined through (6) would be a solution of Problem 6, if the boundary conditions were fulfilled. Using again the jump conditions for the layer potentials, we can show the following lemma.

Lemma 7 *Let $a \in T_d^{0\alpha}(\Gamma)$, $\lambda \in C_*^{0\alpha}(\Gamma)$. The corresponding fields $H^\pm$, E^- defined by (6) are a solution of Problem 6, if and only if a and λ solve*

$$A\begin{pmatrix} a \\ \lambda \end{pmatrix} = -2\begin{pmatrix} c \\ g \end{pmatrix}, \qquad A = \begin{pmatrix} M_k - I & R_k - R_0 + P_k D_k \\ \mu^- N_k & \mu^-(K_k' + I + Q_k D_k) - \mu^+(K_0' - I) \end{pmatrix}.$$

The solvability of this operator equation is stated in the next theorem.

Theorem 1 $A = A_1 + A_2$, $A_1, A_2 : T_d^{0\alpha}(\Gamma) \times C_*^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma) \times C_*^{0\alpha}(\Gamma)$,

$$A_1 = \begin{pmatrix} -I & 0 \\ \mu^- N_k & (\mu^+ + \mu^-)I \end{pmatrix}, \qquad A_2 = \begin{pmatrix} M_k & R_k - R_0 + P_k D_k \\ 0 & \mu^-(K_k' + Q_k D_k) - \mu^+ K_0' \end{pmatrix}.$$

A_1 has a bounded inverse, A_2 is compact. Moreover, A is injective and thus continuously invertible.

Proof: First let us show, that A_1, A_2 really map $T_d^{0\alpha}(\Gamma) \times C_*^{0\alpha}(\Gamma)$ into itself. Since for $a \in T_d^{0\alpha}(\Gamma)$ we deduce

$$\int_{\Gamma_j} (N_k a)(x)ds(x) = 2\int_{\Gamma_j}\int_\Gamma n(x) \cdot \operatorname{curl}_x \left(a(y)\Phi_k(x,y)\right) ds(y)ds(x) = 0$$

by Stokes' Theorem, this is obvious for A_1.

Now let us take a look at A_2. For all $j \in \{1, \ldots, m\}$ it is shown in [1], that

$$1_{\Gamma_j}(x) = \begin{cases} 1 & x \in \Gamma_j \\ 0 & \text{else} \end{cases} \in C^{0\alpha}(\Gamma)$$

is contained in the nullspace $N(I + K_0)$, so that

$$\int_{\Gamma_j} K_0'\lambda\, ds = < K_0'\lambda, 1_{\Gamma_j} > = < \lambda, K_0(1_{\Gamma_j}) > \;=\; < \lambda, -1_{\Gamma_j} > = -\int_{\Gamma_j} \lambda\, ds,$$

since K_0 und K_0' are adjoint with respect to $< u, v > = \int_\Gamma uv\, ds$. But then K_0' maps $C_*^{0\alpha}(\Gamma)$ into itself.

Moreover? div $H^-(x) = 0$ in D^- so that

$$0 = \int_{D_j^-} \text{div } H^- dv = \int_{\Gamma_j} n \cdot H^- ds.$$

Using again the jump relations and the above defined operators we get

$$2\, n \cdot H^- |_\Gamma = (I + K_k' + Q_k D_k)\lambda + N_k a.$$

But $N_k : T_d^{0\alpha}(\Gamma) \to C_*^{0\alpha}(\Gamma)$ and thus $K_k' + Q_k D_k$ maps $C_*^{0\alpha}(\Gamma)$ into itself.

A_1 is obviously continuously invertible and A_2 is compact due to the mapping properties (Lemma 5) of its single components (remember that $C_*^{0\alpha}(\Gamma)$ is a closed subspace of $C^{0\alpha}(\Gamma)$),

Now assume $(a, \lambda)^T$ to be a solution of $A(a, \lambda)^T = 0$. Corresponding to Lemma 7, the fields (6) defined via a, λ are solutions of Problem 6 for $c = 0$ and $g = 0$, so that they have to vanish according to Lemma 4. Thus we conclude

$$0 = 2n \cdot H^+ |_\Gamma = (K_0' - I)\lambda.$$

But $N(I - K_0') = \{0\}$ [1], i.e. $\lambda = 0$. Moreover,

$$0 = 2n \wedge H^- |_\Gamma = (M_k - I)a$$

and since $N(I - M_k) = \{0\}$ for $\text{Im}(k) > 0$ [1], we also have $a = 0$ which completes the proof. ∎

A direct consequence of Lemma 4, Lemma 6 and Lemma 7, Theorem 1 is

Theorem 2 *Problem 6 has a unique solution depending continuously on the given data c and g.*

Now we are coming back to our original task of determining solutions of Problem 3 with circulations $h_i = 0$, $i \in \{1, \ldots, p\}$. The only part which is still missing is a suitable electric field E^+. Let us therefore assume that we already have constructed a solution of Problem 6 as indicated above, that is we have solved $A \begin{pmatrix} a \\ \lambda \end{pmatrix} = -2 \begin{pmatrix} c \\ g \end{pmatrix}$ and computed $H^\pm$, E^- via (6). This means

$$H^+(x) = \int_\Gamma \text{grad}_x\, (\lambda(y)\Phi_0(x, y))\, ds(y)$$

with $\lambda \in C_*^{0\alpha}(\Gamma)$, i.e. $\int_{\Gamma_j} \lambda \, ds = 0 \ \forall j \in \{1, \ldots, m\}$. With the help of Lemma 29 from Appendix A, we get the existence of $b \in T_d^{0\alpha}(\Gamma)$ with

$$\text{Div } b = \lambda \quad \text{on } \Gamma, \qquad \|b\|_{d\alpha,\Gamma} \leq c_\alpha \|\lambda\|_{0\alpha,\Gamma}. \tag{7}$$

Now we define E^+ by

$$E^+(x) = i\omega\mu^+ \int_\Gamma \text{curl}_x \, (b(y)\Phi_0(x, y)) \, ds(y). \tag{8}$$

We immediately see, that $(\text{div } E^+)(x) = 0$ in D^+ and

$$\begin{aligned}
\frac{1}{i\omega\mu^+}\left(\text{curl } E^+\right)(x) &= \text{curl}_x \text{curl}_x \int_\Gamma b(y)\Phi_0(x, y) ds(y) \\
&= (\text{grad}_x \text{div}_x - \Delta_x) \int_\Gamma b(y)\Phi_0(x, y) ds(y) \\
&= \text{grad}_x \int_\Gamma (\text{Div } b)(y)\Phi_0(x, y) ds(y) \\
&= \text{grad}_x \int_\Gamma \lambda(y)\Phi_0(x, y) ds(y) \\
&= H^+(x).
\end{aligned}$$

Moreover

$$\begin{aligned}
\int_{\Gamma_j} n(y) E^+(y) ds(y) &= i\omega\mu^+ \int_{\Gamma_j} \int_\Gamma n(y) \text{curl}_y \, (b(z)\Phi_0(y, z)) \, ds(z) ds(y) \\
&= 0.
\end{aligned}$$

by Stokes' theorem. Due to the properties of the layer potential used in its definition, E^+ depends continuously on b resp. λ and thus on the data c and g.

Collecting these results, we are able to extend the last theorem.

Theorem 3 *For any* $c \in T_d^{0\alpha}(\Gamma)$, $g \in C_*^{0\alpha}(\Gamma)$ *the fields* $H^\pm, E^\pm$ *defined through (6) and (8) by the unique solution* $(a, \lambda)^T$ *of* $A \begin{pmatrix} a \\ \lambda \end{pmatrix} = -2 \begin{pmatrix} c \\ g \end{pmatrix}$ *and* b *from (7) solve Problem 3 with* $h_i = 0$, $i \in \{1, \ldots, p\}$. *Moreover*

$$\max\{\|H^\pm\|_{0\alpha,\bar{D}^\pm}, \|E^\pm\|_{0\alpha,\bar{D}^\pm}\} \leq c_\alpha \max\{\|c\|_{d\alpha,\Gamma}, \|g\|_{0\alpha,\Gamma}\},$$

where c_α *is independent of* g *and* c.

Now let us return to the general case $h_i \in \mathbb{C}$, $i \in \{1, \ldots, p\}$. Since Problem 3 is linear, it is enough to show solvability for c = 0, g = 0, and the p different choices $h_{li} = \delta_{li}$, $i, l \in \{1, \ldots, p\}$.

Let $l \in \{1, \ldots, p\}$ be fixed. By Theorem 3, a solution $\tilde{H}_l^\pm, \tilde{E}_l^\pm$ of Problem 3 for $g_l = 0$, $c_l = n \wedge Z_l^+ \in T_d^{0\alpha}(\Gamma)$, $h_{li} = 0$, $i \in \{1, \ldots, p\}$, exists. Now define

$$\left.\begin{aligned} H_l^+ &= \tilde{H}_l^+ + Z_l^+ \\ E_l^+ &= \tilde{E}_l^+ + i\omega\mu^+ F_l^+ \end{aligned}\right. \quad \text{in } D^+ \qquad \left.\begin{aligned} H_l^- &= \tilde{H}_l^- \\ E_l^- &= \tilde{E}_l^- \end{aligned}\right. \quad \text{in } D^-,$$

where $F_l^+ \in C^1(D^+) \cap C^{0\alpha}(\bar{D}^+)$ is a solution of

$$\operatorname{curl} F_l^+ = Z_l^+, \qquad \operatorname{div} F_l^+ = 0 \qquad \text{in } D^+,$$
$$\int_{\Gamma_j} n \cdot F_l^+ ds = 0 \qquad \forall j \in \{1, \ldots, m\},$$
$$F_l^+(x) = o(1) \qquad |x| \to \infty,$$

given by Lemma 27 in Appendix A. By direct calculations we verify, that $H_l^\pm$, $E_l^\pm$ are a solution of Problem 3 with boundary conditions

$$n \wedge H_l^+ - n \wedge H_l^- = n \wedge \tilde{H}_l^+ - n \wedge \tilde{H}_l^- - n \wedge Z_l^+ = 0,$$
$$n \cdot (\mu^+ H_l^+) - n \cdot (\mu^- H_l^-) = n \cdot (\mu^+ \tilde{H}_l^+) - n \cdot (\mu^- \tilde{H}_l^-) + n \cdot (\mu^+ Z_l^+) = 0$$

and circulations

$$h_{li} = \int_{\gamma_i^+} \tau \cdot H_l^+ dl = \int_{\gamma_i^+} \tau \cdot \tilde{H}_l^+ dl + \int_{\gamma_i^+} \tau \cdot Z_l^+ dl = \delta_{li}.$$

'Together with Theorem 3, we obtain the final existence result of this section.

Theorem 4 *Problem 3 possesses a solution $H^\pm, E^\pm$ with*

$$\max\{\|H^\pm\|_{0\alpha,\bar{D}^+}, \|E^\pm\|_{0\alpha,\bar{D}^+}\} \leq c_\alpha \max\{\|c\|_{d\alpha,\Gamma}, \|d\|_{d\alpha,\Gamma}, \|h\|_\infty\}.$$

5 Well-Posedness of Problem 2

Now we consider Problem 2 and show how to construct solutions using the results from the last section.

Lemma 8 *Problem 2 has at most one solution.*

Proof: Lrt us consider a solution $H^\pm$, $E^\pm$ of Problem 2 for $c = d = 0$. H^+ is a harmonic vector field in D^+ with $H^+(x) = o(1)$, $|x| \to \infty$. Thus, H^+ already decays like $O\left(\frac{1}{|x|^2}\right)$, so that we may apply the Gaussian theorem in the exterior domain D^+

$$\begin{aligned}
i\omega\mu^+ \int_{D^+} H^+ \cdot \bar{H}^+ dx &= -\int_{D^+} \operatorname{curl} \bar{H}^+ \cdot E^+ - \bar{H}^+ \cdot \operatorname{curl} E^+ dx \\
&= \int_\Gamma n \cdot (\bar{H}^+ \wedge E^+) ds \\
&= \int_\Gamma n \cdot (\bar{H}^- \wedge E^-) ds \\
&= (\sigma^- + i\omega\varepsilon^-) \int_{D^-} E^- \cdot \bar{E}^- dx - i\omega\mu^- \int_{D^-} H^- \cdot \bar{H}^- dx.
\end{aligned}$$

Considering the real part of the last equation, we get $E^- = 0$ in D^-. Since $i\omega\mu^- H^- = \operatorname{curl} E^-$, H^- and thus also H^+ must vanish. Now E^+ is a solution of

$$\begin{aligned} curl\ E^+ = 0, &\qquad \operatorname{div} E^+ = 0, \\ \int_{\Gamma_j} n \cdot E^+ ds = 0 &\qquad \forall j \in \{1,\dots,m\}, \\ n \wedge E^+ = 0 &\qquad \text{on } \Gamma, \\ E^+(x) = o(1), &\qquad |x| \to \infty \end{aligned}$$

which has to be identical to zero due to Lemma 26 from Appendix A. ∎

To show existence, we start with the following remark.

Lemma 9 *A solution $H^\pm, E^\pm$ of Problem 2 also solves Problem 3 with boundary data $c \in T_d^{0\alpha}(\Gamma)$ and $g = -\frac{1}{i\omega} Div\ d \in C_*^{0\alpha}(\Gamma)$ and some circulations $h \in \mathbb{C}^p$.*

Proof: By Lemma 22 from Appendix A, the surface divergence of $n \wedge E^\pm$ on Γ is given by

$$\operatorname{Div}(n \wedge E^\pm|_\Gamma) = -n \cdot \operatorname{curl} E^\pm|_\Gamma = -i\omega\, n \cdot (\mu^\pm H^\pm)|_\Gamma.$$

Taking the surface divergence of $n \wedge E^+ - n \wedge E^- = d$ on Γ yields $n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) = g$. Obviously, $g \in C^{0\alpha}(\Gamma)$ and since $g = -\frac{1}{i\omega}\operatorname{Div} d$ on any closed surface Γ_j, $j \in \{1,\dots,m\}$, we get $\int_{\Gamma_j} g\, ds = 0$ by Lemma 21 from Appendix A. ∎

In the existence proof we reverse the order of argumentation. We consider the set of solutions of Problem 3 and determine suitable circulations $h \in \mathbb{C}^p$ and a modified E^+, so that we get a solution of Problem 2.

Theorem 5 *Problem 2 has a unique solution $H^\pm, E^\pm$ with*

$$max\{\|H^\pm\|_{0\alpha,D^\pm}, \|E^\pm\|_{0\alpha,D^\pm}\} \le c_\alpha max\{\|c\|_{d\alpha,\Gamma}, \|d\|_{d\alpha,\Gamma}\}.$$

Moreover, the circulations $h = (h_1,\dots,h_p)^T \in \mathbb{C}^p$, $h_i = \int_{\gamma_i^+} t \cdot H^+ dl$ solve a nonsingular linear $p \times p$ system

$$\begin{aligned} Ah &= b, \\ a_{il} &= \int_\Gamma \left(n \wedge (E_l^+ - E_l^-)\right) \cdot Z_i^+ ds, \qquad (9) \\ b_i &= \int_\Gamma \left(d - n \wedge (E_\star^+ - E_\star^-)\right) \cdot Z_i^+ ds, \end{aligned}$$

$i, l \in \{1,\dots,p\}$, where $H_\star^\pm, E_\star^\pm$ resp. $H_l^\pm, E_l^\pm$, $l \in \{1,\dots,p\}$ are arbitrary solutions of Problem 3 to the data c, $g = -\frac{1}{i\omega} Div\ d$, $h_i = 0$ resp. to $c = 0$, $g = 0$, $h_i = \delta_{li}$, $i \in \{1,\dots,p\}$.

Proof: Let $c, d \in T_d^{0\alpha}(\Gamma)$, $h \in \mathbb{C}^p$ be given, define $g = -\frac{1}{i\omega}\operatorname{Div} d \in C_*^{0\alpha}(\Gamma)$ and consider

$$H_h^\pm = H_\star^\pm + \sum_{l=1}^p h_l H_l^\pm, \qquad E_h^\pm = E_\star^\pm + \sum_{l=1}^p h_l E_l^\pm.$$

$H_h^\pm$, $E_h^\pm$ fulfill all conditions of Problem 2 besides $n \wedge E_h^+ - n \wedge E_h^- = d$ on Γ. If we could find a $F_h^+ \in C^1(D^+) \cap C^{0\alpha}(\bar{D}^+)$ with

$$\begin{aligned}
\operatorname{curl} F_h^+ = 0, \qquad \operatorname{div} F_h^+ &= 0 \qquad \text{in } D^+,\\
n \wedge F_h^+ = e_h = d - n \wedge (E_h^+ - E_h^-) &\qquad \text{on } \Gamma,\\
\int_{\Gamma_j} \tau \cdot F_h^+ ds = 0, &\qquad j \in \{1, \ldots, m\},\\
F_h^+(x) = o(1), &\qquad |x| \to \infty,
\end{aligned} \tag{10}$$

then

$$H^\pm = H_h^\pm, \qquad E^+ = E_h^+ + F_h^+ \qquad E^- = E_h^- \tag{11}$$

would solve Problem 2. But according to Lemma 26 from Appendix A, the above problem for F_h^+ is solvable if and only-if

$$\operatorname{Div} e_h = 0, \qquad \int_\Gamma e_h \cdot Z_i^+ ds = 0, \quad i \in \{1, \ldots, p\}.$$

For the surface divergence of e_h we obtain

$$\operatorname{Div} e_h = \operatorname{Div} d + i\omega \left(n \cdot (\mu^+ H_h^+) - n \cdot (\mu^- H_h^-)\right) = 0$$

since $H_h^\pm$, $E_h^\pm$ are solutions of Problem 3 for $g = -\frac{1}{i\omega}\operatorname{Div} d$.

The second solvability condition of (10) is equivalent to the system of linear equations $Ah = b$ from the assumption. If this system is solvable, we can determine the above mentioned F_h^+ and thus obtain the existence of a solution of Problem 2.

Assume we have $\tilde{h} \in \mathbb{C}^p$, with $A\tilde{h} = 0$ and define

$$\tilde{H}^\pm = \sum_{l=1}^{p} \tilde{h}_l H_l^\pm, \qquad \tilde{E}^\pm = \sum_{l=1}^{p} \tilde{h}_l E_l^\pm.$$

Since $H_\star^\pm$, $E_\star^\pm$ are not involved, $\tilde{H}^\pm$, $\tilde{E}^\pm$ are a solution of Problem 3 for homogeneous boundary data c and g, with $\operatorname{Div}\left(n \wedge (\tilde{E}^+ - \tilde{E}^-)\right) = -\frac{1}{i\omega} g = 0$. Moreover, $A\tilde{h} = 0$ just means

$$\int_\Gamma \left(n \wedge (\tilde{E}^+ - \tilde{E}^-)\right) \cdot Z_i^+ ds = 0, \qquad \forall i \in \{1, \ldots, p\},$$

so that by Lemma 26 from Appendix A we can find a solution $\tilde{F}^+$ of (10) to the boundary value $\tilde{e} = -(n \wedge \tilde{E}^+ - n \wedge \tilde{E}^-)|_\Gamma$ and $\tilde{H}^\pm$, $\tilde{E}^+ + \tilde{F}_h^+$, $\tilde{E}^-$ are a solution of Problem 2 for $c = d = 0$ on Γ. Due to the uniqueness result, of Lemma 8, these fields have to vanish identically. But on the other hand, the $\tilde{h}_l$ are given by $\tilde{h}_l = \int_{\gamma_l'} \tau \cdot \tilde{H}^+ dl$, $l \in \{1, \ldots, p\}$, so that, $A\tilde{h} = 0$ means $\tilde{h} = 0$ and A is nonsingular.

By this result we know, that $Ah = b$ is always uniquely solvable, so we can determine the h_l and the field F_h^+ to obtain a solution $H^\pm$, $E^\pm$ of Problem 2 via (11). Since the solution of Problem 2 is unique, the second part of the assumption is shown.

To prove existence and continuous dependence of the solution on the data we **repeat** the above construction using the solutions $H_\star^\pm$, $E_\star^\pm$ resp. $H_l^\pm$, $H_l^\pm$ whose existence is guaranteed by Theorem 3. Since $H_\star^\pm$, $E_\star^\pm$ depend continuously on c and g = $-\frac{1}{i\omega}\mathrm{Div}\ d$, **the** right hand side b of $Ah = b$ and thus the h_l depend continuously on c and d. Moreover, by Lemma 26 from Appendix A

$$\|F_h^+\|_{0\alpha,\bar{D}^+} \le c_\alpha \|e_h\|_{d\alpha,\Gamma},$$

$$e_h = d - \left(n \wedge (E_\star^+ - E_\star^-) + \sum_{l=1}^{p} h_l n \wedge (E_l^+ - E_l^-)\right),$$

so that $H^\pm$, $E^\pm$ depend continuously on c, d. ∎

6 The Integral Operator Equation

Now we **are** going to present the basic idea how to prove convergence of the solutions of Problem 1 to those of Problem 2 if $\varepsilon > 0$ tends to 0. As we will see, this is equivalent to the norm convergence of certain operators linked to interior and exterior boundary value problems.

We start our considerations by splitting the transmission boundary-value problems up into the following interior and exterior boundary-value problems.

Problem 7 **Let** $\varepsilon > 0$ *and* $k_\varepsilon^+ = \sqrt{\omega^2\mu^+\varepsilon}$. *For* $e \in T_d^{0\alpha}(\Gamma)$ *given, find a* **solution** $H_\varepsilon^+, E_\varepsilon^+ \in C^1(D^+) \cap C^{0\alpha}(\bar{D}^+)$ *of*

$$\begin{aligned} curl\ H_\varepsilon^+ &= -i\omega\varepsilon E_\varepsilon^+ \\ curl\ E_\varepsilon^+ &= i\omega\mu^+ H_\varepsilon^+ \end{aligned} \quad in\ D^+, \qquad n \wedge E_\varepsilon^+ = e \quad on\ \Gamma,$$

$$\omega\mu^+ H_\varepsilon^+(x) \wedge \frac{x}{|x|} - k_\varepsilon^+ E_\varepsilon^+(x) = o\left(\frac{1}{|x|}\right) \qquad |x| \to \infty.$$

Problem 8 *For* $e \in T_d^{0\alpha}(\Gamma)$ *given, find a solution* $H_0^+, E_0^+ \in C^1(D^+) \cap C^{0\alpha}(\bar{D}^+)$ *of*

$$\begin{aligned} curl\ H_0^+ &= 0 \\ curl\ E_0^+ &= i\omega\mu^+ H_0^+ \\ div\ E_0^+ &= 0 \end{aligned} \quad in\ D^+, \qquad n \wedge E_0^+ = e \quad on\ \Gamma,$$

$$\int_{\Gamma_j} n \cdot E_0^+ ds = 0 \qquad \forall j \in \{1, \dots, m\},$$

$$H_0^+(x) = o(1), \quad E_0^+(x) = o(1) \qquad |x| \to \infty.$$

Problem 9 *Let* $\varepsilon \ge 0$ *and* $k_\varepsilon^- = \sqrt{\omega^2\mu^-\varepsilon + i\omega\sigma^-\mu^-}$, $Im(k_\varepsilon^-) > 0$. *For* $e \in T_d^{0\alpha}(\Gamma)$ *given, find a solution* $H_\varepsilon^-, E_\varepsilon^- \in C^1(D^-) \cap C^{0\alpha}(\bar{D}^-)$ *of*

$$\begin{aligned} curl\ H_\varepsilon^- &= (\sigma \quad i\omega\varepsilon) E_\varepsilon^- \\ curl\ E_\varepsilon^- &= i\omega\mu^- H_\varepsilon^- \end{aligned} \quad in\ D^- \qquad n \wedge E_\varepsilon^- = e \quad on\ \Gamma.$$

It is well known ([1],[4]), that all these boundary-value problems have unique solutions depending continuously on the data in the sense that for $\varepsilon \geq 0$

$$\|H_\varepsilon^\pm\|_{0\alpha,\bar{D}^\pm}, \|E_\varepsilon^\pm\|_{0\alpha,\bar{D}^\pm} \leq c_\alpha \|e\|_{d\alpha,\Gamma}.$$

Thus, the operators which map the tangential components of the electric field on Γ to those of the magnetic field are well defined.

Definition 3 *Consider a given boundary value $e \in T_d^{0\alpha}(\Gamma)$ and the fields H_ε^+, E_ε^+ which are for $\varepsilon > 0$ the solution of Problem 7, for $\varepsilon = 0$ the solution of Problem 8 and the fields H_ε^-, E_ε^- which uniquely solve Problem 9. Then we define for $\varepsilon \geq 0$ the linear operators $A_\varepsilon^\pm$ on $T_d^{0\alpha}(\Gamma)$ via*

$$A_\varepsilon^\pm e = n \wedge H_\varepsilon^\pm|_\Gamma .$$

Using the continuous dependence of the solutions of Problem 7-9 on the boundary data together with Lemma. 22 from Appendix A and the Lipschitz contnuity of n on Γ, we immediately get the following mapping properties for $A_\varepsilon^\pm$.

Lemma 10 *For $\varepsilon \geq 0$ the operators $A_\varepsilon^\pm : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ are bounded.*

With the help of the last definition, we are now able to derive an operator equation depending on ε, whose solvability is equivalent to the solvability of Problem 1 for $\varepsilon > 0$ resp. Problem 2 for $\varepsilon = 0$.

Theorem 6 *Let $c, d, e_\varepsilon \in T_d^{0\alpha}(\Gamma)$ and H_ε^+, E_ε^+ be the solution of Problem 7 for $\varepsilon > 0$ resp. Problem 8 for $\varepsilon = 0$ to the boundary value e_ε. Moreover, let H_ε^-, E_ε^- be the solution of Problem 9 to the boundary datn $e_\varepsilon - d$. Then $H_\varepsilon^\pm$, $E_\varepsilon^\pm$ solve Problem 1 for $\varepsilon > 0$ resp. Problem 2 for $\varepsilon = 0$, if and only if*

$$\mathcal{L}_\varepsilon e_\varepsilon = f_\varepsilon, \qquad \mathcal{L}_\varepsilon = A_\varepsilon^+ - A_\varepsilon^-, \qquad f_\varepsilon = c - A_\varepsilon^- d. \tag{1.2}$$

Proof: $H_\varepsilon^\pm$, $E_\varepsilon^\pm$ fulfill the differential equations and radiation conditions of Problem 1 resp. Problem 2. It remains to be shown, that the correct boundary conditions are taken on.

According to the assumptions $n \wedge E_\varepsilon^+ - n \wedge E_\varepsilon^- = d$ on Γ. Moreover

$$n \wedge H_\varepsilon^+|_\Gamma = A_\varepsilon^+(n \wedge E_\varepsilon^+|_\Gamma) = A_\varepsilon^+ e_\varepsilon$$
$$n \wedge H_\varepsilon^-|_\Gamma = A_\varepsilon^-(n \wedge E_\varepsilon^-|_\Gamma) = A_\varepsilon^-(e_\varepsilon - d)$$

so that the boundary condition $n \wedge H_\varepsilon^+ - n \wedge H_\varepsilon^- = c$ holds if and only if e_ε solves (12). ■

Using the results about the transmission boundary value problems obtained in section 4 and 5, we can show unique solvability of (12) for all $\varepsilon \geq 0$.

Theorem 7 *$\mathcal{L}_\varepsilon : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ has a bounded inverse for all $\varepsilon \geq 0$.*

Proof: Let $f \in T_d^{0\alpha}(\Gamma)$ and $H_\varepsilon^\pm, E_\varepsilon^\pm$ be the unique solution of Problem 1 for $\varepsilon > 0$ resp. Problem 2 for $\varepsilon = 0$ with boundary data $c = f$ and $d = 0$. According to the last theorem we get $\mathcal{L}_\varepsilon e_\varepsilon = f$ where $e_\varepsilon = n \wedge E_\varepsilon^+|_\Gamma$, so that $\mathcal{L}_\varepsilon$ is surjective. Moreover by the continuous dependence of $H_\varepsilon^\pm, E_\varepsilon^\pm$ on the boundary data we immediately obtain $\|e_\varepsilon\|_{d\alpha,\Gamma} \leq c_\alpha \|f\|_{d\alpha,\Gamma}$.

Now consider $e_\varepsilon \in T_d^{0\alpha}(\Gamma)$ with $\mathcal{L}_\varepsilon e_\varepsilon = 0$ and the solutions $H_\varepsilon^\pm, E_\varepsilon^\pm$, $\varepsilon \geq 0$ of Problem 7 resp. Problem 8 and Problem 9 to the boundary value e_ε. Then $n \wedge E_\varepsilon^+ = n \wedge E_\varepsilon^-$ on Γ and since $\mathcal{L}_\varepsilon = A_\varepsilon^+ - A_\varepsilon^-$, $A_\varepsilon^\pm(n \wedge E_\varepsilon^\pm) = n \wedge H_\varepsilon^\pm$, *we also* have $n \wedge H_\varepsilon^+ = n \wedge H_\varepsilon^-$. So $H_\varepsilon^\pm, E_\varepsilon^\pm$ are the unique solution of Problem 1 for $\varepsilon > 0$ resp. Problem 2 for $\varepsilon = 0$ to the boundary values $c = 0$ and $d = 0$ and thus have to vanish. But this means $e_\varepsilon = n \wedge E_\varepsilon^+|_\Gamma = 0$ and $\mathcal{L}_\varepsilon$ is injective. ∎

7 Representation of $\mathcal{L}_\varepsilon$ and Convergence in the Simply Connected Case

The strategy of the convergence proof is now obvious. Equation (12) is uniquely solvable for all $\varepsilon \geq 0$ and equivalent to the transmission boundary value problems in the sense defined above. If we can show, that for ε tending to 0 the operators $A_\varepsilon^\pm$ converge to $A_0^\pm$ in the norm induced by $T_d^{0\alpha}(\Gamma)$, we can conclude that the solution e_ε of (12) tends to the solution e_0 of $\mathcal{L}_0 e_0 = f_0$. Thus, the tangential components $n \wedge E_\varepsilon^+|_\Gamma$ of the solution $H_\varepsilon^\pm, E_\varepsilon^\pm$ of Problem 1 converge to the tangential components $n \wedge E_0^+|_\Gamma$ of the solution $H_0^\pm$, $E_0^\pm$ of Problem 2 for $\varepsilon = 0$. As we will see below, this is already enough to ensure the convergence of $H_\varepsilon^\pm, E_\varepsilon^\pm$ *to* $H_0^\pm, E_0^\pm$ in $C^{0\alpha}(\bar{D}^\pm)$.

Because the whole problem is reduced to the convergence of $A_\varepsilon^\pm$ to $A_0^\pm$ for ε tending to 0, it is necessary to derive a more explicit form of these operators than the one given in Definition 3. This form is obtained via the following representation results for the solutions of the interior and exterior boundary value problems given above.

Lemma 11 For c $\in T_d^{0\alpha}(\Gamma)$, $x \in D^\pm$ *define*

$$
\begin{aligned}
\left(M_k^\pm c\right)(x) &= \int_\Gamma \operatorname{curl}_x(c(y)\Phi_k(x,y))ds(y),\\
\left(T_k^\pm c\right)(x) &= \int_\Gamma \operatorname{curl}_x \operatorname{curl}_x(c(y)\Phi_k(x,y))ds(y)\\
&= \int_\Gamma \operatorname{Div} c(y) \operatorname{grad}_x \Phi_k(x,y)ds(y) + k^2 \int_\Gamma c(y)\Phi_k(x,y)ds(y).
\end{aligned}
$$

Then $M_k^\pm, T_k^\pm : T_d^{0\alpha}(\Gamma) \to C^{0\alpha}(\bar{D}^\pm)$ *and*

$$\|M_k^\pm c\|_{0\alpha,\bar{D}^\pm}, \|T_k^\pm c\|_{0\alpha,\bar{D}^\pm} \leq c_\alpha \|c\|_{d\alpha,\Gamma},$$

where c_α *is independent of* $c \in T_d^{0\alpha}(\Gamma)$.

Lemma 12 *Let $E_\varepsilon^+, H_\varepsilon^+$ be a solution of Problem 7 for $\varepsilon > 0$ resp. Problem 8 for $\varepsilon = 0$ and $E_\varepsilon^-, H_\varepsilon^-$ be a solution of Problem 9. Then*

$$\begin{array}{rclcl}
M_{k_\varepsilon^+}^+ (n \wedge H_\varepsilon^+|_\Gamma) & + & \frac{1}{i\omega\mu^+} T_{k_\varepsilon^+}^+ (n \wedge E_\varepsilon^+|_\Gamma) & = & H_\varepsilon^+ \text{ in } D^+, \\
M_{k_\varepsilon^-}^- (n \wedge H_\varepsilon^-|_\Gamma) & + & \frac{1}{i\omega\mu^-} T_{k_\varepsilon^-}^- (n \wedge E_\varepsilon^-|_\Gamma) & = & -H_\varepsilon^- \text{ in } D^-, \\
M_{k_\varepsilon^-}^- (n \wedge E_\varepsilon^-|_\Gamma) & + & \frac{1}{\sigma - i\omega\varepsilon} T_{k_\varepsilon^-}^- (n \wedge H_\varepsilon^-|_\Gamma) & = & -E_\varepsilon^- \text{ in } D^-.
\end{array}$$

The proof of Lemma 11 and Lemma 12 is exactly the same as for the standard representation theorems which can be found for example in [1].

Approaching the boundary Γ with x from the exterior resp. the interior domain and using the jump relations for the layer potentials involved as well as the operators of Definition 2, we get

Lemma 13 *Let E_ε^+, H_ε^+ be a solution of Problem 7 for $\varepsilon > 0$ resp. of Problem 8 for $\varepsilon = 0$ and E_ε^-, H_ε^- be a solution of Problem 9. Then*

$$\begin{aligned}
(I - M_{k_\varepsilon^+})(n \wedge H_\varepsilon^+|_\Gamma) &= \frac{1}{i\omega\mu^+} T_{k_\varepsilon^+} (n \wedge E_\varepsilon^+|_\Gamma), \\
(I + M_{k_\varepsilon^-})(n \wedge H_\varepsilon^-|_\Gamma) &= -\frac{1}{i\omega\mu^-} T_{k_\varepsilon^-} (n \wedge E_\varepsilon^-|_\Gamma).
\end{aligned}$$

If the operators $I - M_{k_\varepsilon^+}$ resp. $I + M_{k_\varepsilon^-}$ were continuously invertible for all $\varepsilon \geq 0$, we would directly obtain explicit representations for $A_\varepsilon^\pm$. According to Lemma 23 from Appendix A, this is true for $I + M_{k_\varepsilon^-}$ since $\operatorname{Im}(k_\varepsilon^-) > 0$. For topological genus $p \geq 1$ we do not get the desired representation for $I - M_{k_\varepsilon^+}$ since $I - M_k$ is singularly perturbed for $k \to 0$.

Theorem 8 *For the operator A_ε^- holds*

$$A_\varepsilon^- = -\frac{1}{i\omega\mu^-}(I + M_{k_\varepsilon^-})^{-1} T_{k_\varepsilon^-} \qquad 0 \leq \varepsilon. \tag{13}$$

Depending on the domains $D^\pm$ there exists a constant $\delta > 0$, so that

$$A_\varepsilon^+ = \frac{1}{i\omega\mu^+}(I - M_{k_\varepsilon^+})^{-1} T_{k_\varepsilon^+}, \qquad 0 < \varepsilon \leq \delta. \tag{14}$$

Moreover

$$\lim_{\varepsilon \downarrow 0} \|A_\varepsilon^- - A_0^-\| = 0$$

in the operator norm induced by $T_d^{0\alpha}(\Gamma)$.

If $D^\pm$ are simply connected, the representation (14) also holds for $\varepsilon = 0$ and

$$\lim_{\varepsilon \downarrow 0} \|A_\varepsilon^+ - A_0^+\| = 0$$

in the operator norm induced by $T_d^{0\alpha}(\Gamma)$.

Proof: The results on the representations (13),(14) are obtained as described above. The convergence results are then a consequence of Lemma 34 from Appendix B. ■

Corollary 1 *Let $D^{\pm}$ be simply connected. Then*

$$\lim_{\varepsilon \downarrow 0} \|\mathcal{L}_\varepsilon - \mathcal{L}_0\| = 0$$

in the operator norm induced by $T_d^{0\alpha}(\Gamma)$.

8 Singular Perturbations and the Multiply Connected Case

Now we have to overcome the difficulty that (14) is not valid for $\varepsilon = 0$ if D^* are multiply connected. We start our considerations of this case by modifying some general results on singular perturbation problems from [1] so that they apply to our special situation.

Theorem 9 *Let X, Y be Banuch-spaces $< \cdot, \cdot >: X \times Y \to \mathbb{C}$ a nondegenerate bilinear form and $K \subset \mathbb{C}$ be a subset of complex numbers with accumulation point $0 \in K$. Consider a family of compact linear operators $\{A_k : X \to X | k \in K\}$ and their adjoints $A'_k : Y \to Y$ with respect to $< \cdot, \cdot >$. Assume the operators $L_k = I - A_k$ to be injective for $k \neq 0$, the Riesz-number of L_0 to be one and*

$$A_k = A_0 + Ck^2 + D(k) \tag{15}$$

where $C : X \to X$ is a bounded linear operator independent of k and $D(k) : X \to X$ is linear with $\| D(k)\| = o(k^2)$ for $k \to 0$.

Moreover let $\{H_k : X \to X | k \in K\}$ be a family of bounded linear operators with $\lim_{k\to 0} \|H_k - H_0\| = 0$, $\{b_1, \ldots, b_p\}$ be a basis of the nullspace $N(L'_0)$, $L'_0 = I - A'_0$, and assume

$$\begin{gathered} < H_k q, b_i >= g_i(q)k^2 + h_i(q, k), \\ |g_i(q)| \leq c_i\|q\|, \qquad |h_i(q, k)| \leq d_i(k)\|q\|, \qquad d_i(k) = o(k^2), \qquad k \to 0 \end{gathered} \tag{16}$$

for all $i \in \{1, \ldots, p\}$, where the constants c_i are independent of k and $q \in X$ and the functions $d_i(k)$ are independent of $q \in X$.

For $q \in X$ fixed, there exists a $\phi_0 \in X$ with

$$L_0\phi_0 = H_0 q \tag{17}$$

so that for the unique solution ϕ_k, $k \neq 0$ of

$$L_k\phi_k = H_k q, \tag{18}$$

holds

$$\|\phi_k - \phi_0\| \leq d(k)\|q\|, \qquad d(k) = o(1), \qquad k \to 0,$$

where $d(k)$ is independent of q.

Proof: In a first step we reduce the solution of (18) (which is in general posed in an infinite dimensional space X) to an equation posed in the finite dimensional nullspace $N(L_0)$.

Since the Riesz-number of L_0 is 1, we have $X = N(L_0) \oplus L_0(X)$ and define the projection operator

$$P : X = N(L_0) \oplus L''(X) \to N(L_0). \tag{19}$$

According to [1] the linear operator

$$L_0^+ = (L_0 \quad P)^{-1}(I - P),$$

is bounded. Furthermore we define

$$Q_k = L_0^+(L_0 - L_k), \qquad F_k^+ = (I - Q_k)^{-1} L_0^+.$$

The existence and continuity of $(I - Q_k)^{-1}$ f0llows from (15) and a Neumann series expansion for sufficiently small $|k|$. From now on we assume without loss of generality that $I - Q_k$ is boundedly invertible for all $k \in K$. A straightforward calculation using the above introduced operators shows that

$$\phi_k = \psi_k + (I - Q_k)^{-1}\chi_k, \qquad \psi_k = \begin{cases} F_k^+ H_k q & k \neq 0 \\ L_0^+ H_0 q & k = 0 \end{cases}, \qquad \chi_k \in N(L_0). \tag{20}$$

For $k \neq 0$, ϕ_k is a solution of (18) if and only if $\chi_k \in N(L_0)$ is a solution of

$$PL_k(I - Q_k)^{-1}\chi_k = PH_k q - PL_k\psi_k \tag{21}$$

(for details see [1], Chapter 1). By (20)

$$\phi_k - \phi_0 = \psi_k - \psi_0 + (I - Q_k)^{-1}\chi_k - \chi_0. \tag{23}$$

To show convergence of ϕ_k to ϕ_0 we treat both parts of the right hand side of (22) separately.

Using (30) we find

$$\psi_k - \psi_0 = (F_k^+ - L_0^+)H_k q + L_0^+(H_k - H_0)q.$$

Therefore

$$||\psi_k - \psi_0|| \leq ||F_k^+ - L_0^+||\,||H_k||\,||q|| + ||L_0^+||\,||H_k - H_0||\,||q||.$$

But $\lim_{k\to 0} ||F_k^+ - L_0^+|| = 0$ and the assumption $\lim_{k\to 0} ||H_k - H_0|| = 0$ imply

$$||\psi_k - \psi_0|| \leq e_1(k)||q||, \qquad e_1(k) = o(1) \qquad k \to 0, \tag{23}$$

where $e_1(k)$ is independent of q.

In the next step we prove a similar estimate for $||\chi_k - \chi_0||$. Considering the left hand side of (21) we obtain by a straightforward calculation using (15) and $L_k = I - A_k$

$$PL_k(I - Q_k)^{-1}|_{N(L_0)} = -\left(k^2 PC|_{N(L_0)} + H(k)|_{N(L_0)}\right), \tag{24}$$

$$H(k)_{N(L_0)} = PD(k)|_{N(L_0)} + \left(PCk^2 + PD(k)\right) Q_k(I - Q_k)^{-1}|_{N(L_0)}.$$

Because of $\lim_{k\to 0} \|Q_k\| = 0$ and $\|D(k)\| = o(k^2)$, $k \to 0$, we conclude that $H(k)$ satisfies

$$\|H(k)\| = o(k^2), \qquad k \to 0.$$

Let $\chi_k \in N(L_0)$, $k \neq 0$ be a solution of $PL_k(I - Q_k)^{-1}\chi_k = 0$, i.e. χ_k solves (21) with $q = 0$. With the equivalence of (18) and (21) for $k \neq 0$, we conclude that ϕ_k is a solution of $L_k\phi_k = 0$, $k \neq 0$, i.e. by uniqueness $\phi_k = 0$. Since $q = 0$ we get using (20)

$$\chi_k = (I - Q_k)(\phi_k - F_k^+ H_k q) = 0.$$

So we know that the mapping $PL_k(I - Q_k)^{-1}|_{N(L_0)}$ is invertible and by (24)

$$\left(PL_k(I - Q_k)^{-1}|_{N(L_0)}\right)^{-1} = -k^{-2}\left(PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)}\right)^{-1} \tag{25}$$

where

$$k(k) = k^{-2}H(k), \qquad \|\tilde{H}(k)\| = o(1), \qquad k \to 0. \tag{26}$$

Now we consider the right hand side of (21). Given the basis $\{b_1, \ldots, b_p\}$ of $N(L_0')$ we find a basis $\{a_1, \ldots, a_p\}$ of $N(L_0)$ such that

$$< a_i, b_j >= \delta_{ij} \qquad \forall i, j \in \{1, \ldots, p\}.$$

The projector (19) is given as $Pq = \sum_{i=1}^{p} < q, b_i > a_i$, so that by (16)

$$PH_kq = \sum_{i=1}^{p} < H_kq, b_i > a_i = \sum_{i=1}^{p} g_i(q)k^2 a_i + \sum_{i=1}^{p} h_i(q, k)a_i = g(q)k^2 + h(q, k) \tag{27}$$

with $g(q), h(q, k) \in N(L_0)$, $g(q)$ being independent of k. Obviously

$$\|g(q)\| \leq c\|q\|, \qquad \|h(q, k)\| \leq e_2(k)\|q\|, \qquad e_2(k) = o(k^2), \qquad k \to 0 \tag{28}$$

with c being independent of q and k, $e_2(k)$ being independent of q.

Using (27) and (20) a straightforward calculation implies

$$\begin{aligned} PH_kq - PL_k\psi_k &= (g(q) + PC\psi_0 + G(q, k))\, k^2, \\ G(q, k) &= PC(F_k^+H_k - L_0^+H_0)q + k^{-2}PD(k)(F_k^+H_kq) + k^{-2}h(q, k). \end{aligned} \tag{29}$$

By $\lim_{k\to 0} \|F_k^+ - L_0^+\| = 0$, $\lim_{k\to 0} \|H_k - H_0\| = 0$, the assumption $\|D(k)\| = o(k^2)$, $k \to 0$, and (28) we obtain

$$\|G(q, k)\| \leq e_3(k)\|q\|, \qquad e_3(k) = o(1), \qquad k \to 0, \tag{30}$$

and $e_3(k)$ being independent of q.

With these results we are able to consider $\chi_k - \chi_0$. According to (21),(25),(29)

$$\begin{aligned} \chi_k &= \left(PL_k(I - Q_k)^{-1}|_{N(L_0)}\right)^{-1} (PH_kq - PL_k\psi_k) \\ &= -\left(PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)}\right)^{-1} (g(q) + PC\psi_0 + G(q, k)). \end{aligned}$$

Following [1] the mapping $PC|_{N(L_0)}$ is continuously invertible. Let

$$\chi_0 = -\left(PC|_{N(L_0)}\right)^{-1}(g(q) + PC\psi_0)$$

i.e.

$$\begin{aligned}\chi_k - \chi_0 &= -\left(\left(PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)}\right)^{-1} - \left(PC|_{N(L_0)}\right)^{-1}\right)(g(q) + PC\psi_0) \\ &\quad - \left(PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)}\right)^{-1} G(q, k).\end{aligned}$$

But (28) ensures $||g(q)|| \leq c||q||$ and therefore

$$||g(q) + PC\psi_0|| = ||g(q) + PCL_0^+ H_0 q|| \leq c||q||,$$

with c being independent of q. Using the relations (26) and (30) we obtain

$$||\chi_k - \chi_0|| \leq e_4(k)||q||, \qquad e_4(k) = o(1), \qquad k \to 0, \tag{31}$$

$e_4(k)$ being independent of q.

Collecting all results we see, that (22) provides

$$||\phi_k - \phi_0|| \leq ||\psi_k - \psi_0|| + ||(I - Q_k)^{-1}||\,||\chi_k - \chi_0|| + ||(I - Q_k)^{-1} - I||\,||\chi_0||.$$

Using $\lim_{k \to 0} ||Q_k|| = 0$, (23) and (31) we draw the conclusion

$$||\phi_k - \phi_0|| = d(k)||q||,$$

where $d(k)$ is independent of q, $d(k) = o(1)$ as $k \to 0$. Finally ϕ_0 solves (17) since

$$\begin{aligned}||L_0\phi_0 - H_0 q|| &= ||L_0(\phi_0 - \phi_k) + (L_0 - L_k)\phi_k + L_k\phi_k - H_0 q|| \\ &\leq ||L_0||\,||\phi_k - \phi_0|| + ||L_k - L_0||\,||\phi_k|| + ||H_k - H_0||\,||q||.\end{aligned}$$

∎

In order to prove uniform Hölder convergence of the operators A_ε^+ we apply the last theorem to

$$(I - M_{k_\varepsilon^+})(n \wedge H_\varepsilon^+) = \frac{1}{i\omega\mu^+} T_{k_\varepsilon^+}(n \wedge E_\varepsilon^+), \qquad \varepsilon \to 0$$

(see Lemma 13 and Theorem 8). It remains to be checked that the operators $M_{k_\varepsilon^+}$ and $T_{k_\varepsilon^+}$ fulfill all the assumptions of Theorem 9.

Since $k_\varepsilon^+ = \sqrt{\omega^2\mu^+\varepsilon}$, the limit $\varepsilon \to 0$ is equivalent to $k_\varepsilon^+ \to 0$. To simplify the notation, we use k instead of k_ε^+ if there is no danger of confusion.

Lemma 14 *Let $K = \{t \mid t \in \mathbb{C},\ Im(t) \geq 0,\ |t| < \delta\}$, δ small enough. Then $M_k : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$, $k \in K$, is a family of compact operators with adjoint $M_k' : T^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma)$ with respect to the dual system (5). $L_k = I - M_k$ is* *continuously invertible on $T_d^{0\alpha}(\Gamma)$ for all $k \in K$, $k \neq 0$. Moreover the Riesz-number of L_0 is one and*

$$M_k = M_0 + Ck^2 + D(k),$$

where $C : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ is a bounded linear mapping independent of k and $D(k) : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ is linear such that

$$\|D(k)\| = o(k^2), \qquad k \to 0.$$

Proof: Using Lemma 5, Lemma 23 from Appendix A, the definition of the set K and Fredholm's alternative, the first part is obvious. The result concerning the Riesz-number of L_0 is found in [1].

To check the asymptotic behaviour of M_k we consider an arbitrary $q \in T_d^{0\alpha}(\Gamma)$ and decompose

$$\begin{aligned}(M_k q)(x) - (M_0 q)(x) &= 2n(x) \wedge \int_\Gamma \operatorname{curl}_x \left(q(y)\left(\Phi_k(x,y) - \Phi_0(x,y)\right)\right) ds(y) \\ &= k^2 (Cq)(x) + (D(k)q)(x),\end{aligned}$$

where

$$\begin{aligned}(Cq)(x) &= -\frac{1}{4\pi} n(x) \wedge \int_\Gamma \operatorname{curl}_x \left(q(y)\,|x-y|\right) ds(y), \\ (D(k)q)(x) &= 2n(x) \wedge \int_\Gamma \operatorname{curl}_x \left(q(y)\left(\Phi_k(x,y) - \Phi_0(x,y) + \frac{k^2}{8\pi}|x-y|\right)\right) ds(y).\end{aligned}$$

To prove the assertions concerning the operators C and D we use the same techniques as [1] to examine surface potentials. Therefore we concentrate on the essential steps.

The operator C can be written as

$$(Cq)(x) = \frac{1}{4\pi} n(x) \wedge \int_\Gamma q(y) \wedge \frac{x-y}{|x-y|} ds(y).$$

Due to the boundedness of Γ we obtain for the surface potential C with kernel $K(x,y) = \frac{x-y}{|x-y|}$ the mapping property

$$C : T_d^{0\alpha}(\Gamma) \to T^{0,\alpha}(\Gamma), \qquad \|Cq\|_{0\alpha,\Gamma} \leq c_1 \|q\|_\infty,$$

where c_1 is independent of q. To check the surface divergence of Cq, we define

$$E(x) = \frac{1}{4\pi} \operatorname{curl}_x \int_\Gamma \operatorname{curl}_x (q(y)|x-y|) ds(y), \qquad x \in D^-.$$

Using the identity $\operatorname{curl}_x \operatorname{curl}_x = \operatorname{grad}_x \operatorname{div}_x - \Delta_x$ and Lemma 21 from Appendix A one immediately arrives at

$$E(x) = \frac{1}{4\pi} \int_\Gamma (\operatorname{Div} q)(y) \frac{x-y}{|x-y|} ds(y) - \frac{1}{4\pi} \int_\Gamma q(y) \frac{2}{|x-y|} ds(y)$$

for all $x \in D^-$. In complete analogy with [1] we can extend $E(x)$ in a Hölder continuous fashion up to the boundary Γ such that $\|E\|_{0\alpha,\Gamma} \leq c_2\|q\|_{d\alpha,\Gamma}$ with c_2 being independent of q. Lemma 22 provides Div $Cq = -n \cdot E|_\Gamma$ and the Lipschitz continuity of the unit outward normal n on Γ leads to

$$\|\text{Div } Cq\|_{0\alpha,\Gamma} \leq c_3\|q\|_{d\alpha,\Gamma},$$

c_3 independent of q. Therefore

$$C : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma), \qquad \|Cq\|_{d\alpha,\Gamma} \leq c_4\|q\|_{d\alpha,\Gamma},$$

where c_4 is independent of q.

It remains to consider the operator $D(k)$. For $q \in T_d^{0\alpha}(\Gamma)$ we can write

$$(D(k)q)(x) = -2n(x) \wedge \int_\Gamma q(y) \wedge G(x,y)ds(y),$$

with

$$G(x, y) = \text{grad}_x \left(\Phi_k(x,y) - \Phi_0(x,y) + \frac{k^2}{8\pi}|x-y|\right). \tag{32}$$

Using a power series expansion of $\Phi_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ we obtain

$$G(x,y) = \frac{x-y}{4\pi}(ik)^3 \sum_{n=0}^{\infty} \frac{(ik|x-y|)^n}{(n+1)!(n+3)}.$$

Due to the boundedness of Γ we find a ball B_d such that $\Gamma \subset B_d$. With the power series representation of the kernel $G(x,y)$ we get for arbitrary $x_1, x_2 \in B_d, y \in \Gamma$ with $|x_1 - x_2| \leq \frac{1}{2}|x_1 - y|$ the estimate

$$|G(x_1,y) - G(x_2,y)| \leq 2\frac{|x_1-x_2|}{4\pi}|k^3|e^{2ds} \leq c_5|k^3|\,|x_1-x_2|,$$

where $s = \omega\sqrt{\delta\mu^+}$ is an upper bound for $|k|$, $k \in K$, and c_5 is independent of k, x_1, x_2 und y. According to [1] we get

$$\frac{1}{k^3}D(k) : T_d^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma), \qquad \|\frac{1}{k^3}D(k)q\|_{0\alpha,\Gamma} \leq c_6\|q\|_\infty,$$

c_6 independent of k and q. In order to examine the surface divergence we consider for $x \in D^-$

$$\begin{aligned} F(x) &= 2\,\text{curl}_x \int_\Gamma \text{curl}_x \left(q(y)\left(\Phi_k(x,y) - \Phi_0(x,y) + \frac{k^2}{8\pi}|x-y|\right)\right) ds(y) \\ &= 2\int_\Gamma (\text{Div } q)(y)\, G(x,y)ds(y) + 2\int_\Gamma q(y)H(x,y)ds(y), \end{aligned}$$

where $G(x,y)$ was defined in (32) and $H(x,y) = k^2(\Phi_k(x,y) - \Phi_0(x,y))$. Because of $|H(x,y)| \leq Mk^3$ the potential F can be extended in a Hölder continuous fashion up to the boundary I'. Lernma 22 from Appendix A provides Div $D(k)q = -n \cdot F|_\Gamma$ and from the continuous dependence of the layer potentials on their densities we obtain

$$||\text{Div } D(k)q||_{0\alpha,\Gamma} \leq c_7|k^3|(||q||_{0\alpha,\Gamma} + ||\text{Div } q||_{0\alpha,\Gamma}).$$

Thus

$$\frac{1}{k^3}D(k) : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma), \qquad ||\frac{1}{k^3}D(k)q||_{d\alpha,\Gamma} \leq c_8||q||_{d\alpha,\Gamma},$$

where c_8 is independent of k and q. ∎

Lemma 15 *The linear operators $T_k : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ from* **Definition 2** *are bounded with $\lim_{k\to 0} ||T_k - T_0|| = 0$. $Z_i^-|_\Gamma$, $i \in \{$ **1,. . .** $,p\}$ are* **a** *basis of $N(I - M_0')$ and for arbitrary $q \in T_d^{0\alpha}(\Gamma)$ holds*

$$< T_kq, Z_i^- >= g_i(q)k^2 + h_i(q, k) \qquad i \in \{I,. . . ,p\},$$

where *g;(q) is independent of k for all $i \in \{ 1,. . . ,p\}$.* Moreover

$$|g_i(q)| \leq c_i||q||_{d\alpha,\Gamma}, \qquad |h_i(q, k)| \leq d_i(k)||q||_{d\alpha,\Gamma}, \qquad d;(k) = o(k^2), \qquad k \to 0,$$

$\forall i \in \{1,\dots,p\}$ with c_i being independent of k and q resp. $d_i(k)$ is independent of q.

Proof: The results about the convergence of the T_k resp. the basis of $N(I - M_0')$ are shown in Lemma 34 in Appendix B resp. Lemma 23 in Appendix A.

Consider T_k and n A Z_i^- on Γ, which lies in $T_d^{0\alpha}(\Gamma)$ since Div $(n$ A $Z_i^-) = -n \cdot$ curl $Z_i^- = 0$ on Γ by Lemma 22 from Appendix A. For arbitrary $i \in \{ 1, . . . ,$ p$\}$ and $q \in T_d^{0\alpha}(\Gamma)$ holds due to Lemma 5

$$\begin{aligned} < T_kq, Z_i^- > &= - < T_kq, (n \wedge (n \wedge Z_i^-)) > \\ &= - < n \wedge q, T_k(n \wedge Z_i^-) > \\ &= < q, n \text{ A } T_k(n \text{ A } Z_i^-)>, \end{aligned} \tag{33}$$

$< \cdot, . >$ being the bilinear form of the dual system (5). But

$$\begin{aligned}\left(T_k(n \text{ A } Z_i^-)\right)(x) = \ & 2n(x) \text{ A} \int_\Gamma \text{grad}_x (\text{Div } (n(y) \text{ A } Z_i^-(y))\Phi_k(x, y)\big) ds(y) \\ & +2k^2n(x) \text{ A} \int_\Gamma \text{n(y) A } Z_i^-(y)\Phi_k(x, y)ds(y).\end{aligned}$$

Using again Div *(n* **A** $Z_i^-) = 0$ on Γ we get

$$\begin{aligned}\left(T_k(n \text{ A } Z_i^-)\right)(x) = \ & 2k^2n(x) \text{ A} \int_\Gamma \text{n(y) A } Z_i^-(y)\Phi_0(x, y)ds(y) \\ & +2k^2n(x) \text{ A} \int_\Gamma \text{n(y) A } Z_i^-(y)(\Phi_k(x, y) - \Phi_0(x, y))ds(y).\end{aligned}$$

According to (33) we find a decomposition

$$< T_kq, Z_i^- >= g_i(q)\, k^2 + h_i(q, k),$$

with

$$g_i(q) = < q, 2n \wedge \left(n \wedge \int_\Gamma n \wedge Z_i^- \Phi_0 ds \right) >$$

$$h_i(q,k) = < q, 2k^2 n \wedge \left(n \wedge \int_\Gamma n \wedge Z_i^- (\Phi_k - \Phi_0) ds \right) > .$$

The continuity of the bilinear form $< \cdot, \cdot >$ provides

$$|g_i(q)| \le c_i \|q\|_{d\alpha,\Gamma}, \qquad |h_i(q,k)| \le d_i(k) \|q\|_{d\alpha,\Gamma}, \qquad d_i(k) = o(k^2), \qquad k \to 0$$

with c_i being independent of k and q resp. $d_i(k)$ being independent of q. ∎

Now we come to the final result of this section.

Theorem 10 *Let $A_\varepsilon^+ : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ be the operators of Definition 3. Then*

$$\lim_{\varepsilon \downarrow 0} \|A_\varepsilon^+ - A_0^+\| = 0$$

in the induced operator norm.

Proof: In Lemma 14 and Lemma 15 we have shown that, for arbitrary $q \in T_d^{0\alpha}(\Gamma)$

$$(I - M_{k_\varepsilon^+}) \phi_{k_\varepsilon^+} = \frac{1}{i\omega\mu^+} T_{k_\varepsilon^+} q, \qquad k_\varepsilon^+ = \sqrt{\omega^2 \mu^+ \varepsilon} \in K, \qquad \varepsilon \to 0, \tag{34}$$

fits into the frame of the singular pertubation problems considered in Theorem 9. For $k_\varepsilon^+ \in K$, $k_\varepsilon^+ \neq 0$ we obtain from Theorem 8 and (34)

$$\phi_{k_\varepsilon^+} = A_\varepsilon^+ q.$$

Given an arbitrary $q \in T_d^{0\alpha}(\Gamma)$, Theorem 9 provides a unique ϕ_0 which fulfills

$$(I - M_0)\phi_0 = \frac{1}{i\omega\mu^+} T_0 q, \qquad \|\phi_{k_\varepsilon^+} - \phi_0\|_{d\alpha,\Gamma} \le d(k_\varepsilon^+) \|q\|_{d\alpha,\Gamma},$$

where $d(k_\varepsilon^+)$ is independent of q, $d(k_\varepsilon^+) = o(1)$ as $k_\varepsilon^+ \to 0$. Defining $\tilde{A}_0^+$ by $\tilde{A}_0^+ q = \phi_0$ we obtain

$$\begin{aligned} \|A_\varepsilon^+ - \tilde{A}_0^+\| &= \sup_{q \in T_d^{0\alpha}(\Gamma), q \neq 0} \frac{\|A_\varepsilon^+ q - \tilde{A}_0^+ q\|_{d\alpha,\Gamma}}{\|q\|_{d\alpha,\Gamma}} \\ &= \sup_{q \in T_d^{0\alpha}(\Gamma), q \neq 0} \frac{\|\phi_{k_\varepsilon^+} - \phi_0\|_{d\alpha,\Gamma}}{\|q\|_{d\alpha,\Gamma}} = d(k_\varepsilon^+). \end{aligned} \tag{35}$$

This shows that the bounded linear operators A_ε^+ converge uniformly for $\varepsilon \to 0$ to the bounded linear operator $\tilde{A}_0^+$.

By the definition of A_ε^+, $\varepsilon > 0$ we know that $A_\varepsilon^+ q = n \wedge H_\varepsilon^+|_\Gamma$, $H_\varepsilon^+, E_\varepsilon^+$ being the unique solution of Problem 7 to the boundary value $q \in T_d^{0\alpha}(\Gamma)$. But E_ε^+ also solves

$$\begin{aligned} \triangle E_\varepsilon^+ + (k_\varepsilon^+)^2 E_\varepsilon^+ = 0 \qquad & \text{in } D^+ \\ \operatorname{div} E_\varepsilon^+ = 0, \quad n \wedge E_\varepsilon^+ = q \qquad & \text{on } \Gamma, \end{aligned}$$

with radiation condition

$$\operatorname{curl} E_\varepsilon^+(x) \wedge \frac{x}{|x|} - ik_\varepsilon^+ E_\varepsilon^+(x) \;=\; o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty.$$

Let H_0^+, E_0^+ be the unique solution of Problem 8 to the boundary value $q \in T_d^{0\alpha}(\Gamma)$. Theorem 5.9 from [1] and curl $E_0^+ = i\omega\mu^+ H_0^+$ lead to

$$\lim_{\varepsilon\downarrow 0} \|n \wedge H_\varepsilon^+ - n \wedge H_0^+\|_{d\alpha,\Gamma} = 0. \tag{36}$$

Since $n \wedge H_\varepsilon^\pm|_\Gamma = A_\varepsilon^\pm q$ for $\varepsilon \geq 0$ we finally get

$$\begin{aligned}
\left\|\left(A_0^+ - \tilde{A}_0^+\right) q\right\|_{d\alpha,\Gamma} &= \left\|\left(A_\varepsilon^+ - A_0^+\right) q\right\|_{d\alpha,\Gamma} + \left\|\left(A_\varepsilon^+ - \tilde{A}_0^+\right) q\right\|_{d\alpha,\Gamma} \\
&\leq \|n \wedge H_\varepsilon^+ - n \wedge H_0^+\|_{d\alpha,\Gamma} + \|A_\varepsilon^+ - \tilde{A}_0^+\| \, \|q\|_{d\alpha,\Gamma}
\end{aligned}$$

and by (35),(36) A_0^+ and $\tilde{A}_0^+$ coincide so that A_ε^+ converges in norm to A_0^+. ■

Corollary 2 *Let $D^\pm$ be of topological genus $p \geq 0$ and $\mathcal{L}_\varepsilon$, $\varepsilon \geq 0$ the operators introduced in Theorem 6. Then*

$$\lim_{\varepsilon\downarrow 0} \|\mathcal{L}_\varepsilon - \mathcal{L}_0\| = 0$$

in the operator norm induced by $T_d^{0\alpha}(\Gamma)$.

9 Convergence of the Electromagnetic Fields

Before showing convergence of the corresponding fields in the domains $D^\pm$ we focus on their tangential components on the boundary Γ.

Lemma 16 *Consider sequences $c_\varepsilon, d_\varepsilon \in T_d^{0\alpha}(\Gamma)$, $\varepsilon > 0$ such that*

$$\lim_{\varepsilon\downarrow 0} \|c_\varepsilon - c_0\|_{d\alpha,\Gamma} = 0, \qquad \lim_{\varepsilon\downarrow 0} \|d_\varepsilon - d_0\|_{d\alpha,\Gamma} = 0.$$

Let $H_\varepsilon^\pm$, $E_\varepsilon^\pm$ be the solution of Problem 1 for $\varepsilon > 0$ to the boundary inhomogeneities c_ε, $d_\varepsilon \in T_d^{0\alpha}(\Gamma)$. Moreover, let $H_0^\pm$, $E_0^\pm$ be the solution of Problem 2 with the boundary inhomogeneities c_0, $d_0 \in T_d^{0\alpha}(\Gamma)$. Then

$$\lim_{\varepsilon\downarrow 0} \|n \wedge H_\varepsilon^\pm - n \wedge H_0^\pm\|_{d\alpha,\Gamma} = 0, \qquad \lim_{\varepsilon\downarrow 0} \|n \wedge E_\varepsilon^\pm - n \wedge E_0^\pm\|_{d\alpha,\Gamma} = 0.$$

Proof: Assume $0 \leq \varepsilon < \delta$ and define $e_\varepsilon = n \wedge E_\varepsilon^+|_\Gamma$. According to Theorem 6 we get $\mathcal{L}_\varepsilon e_\varepsilon = f_\varepsilon$, $\mathcal{L}_\varepsilon = A_\varepsilon^+ - A_\varepsilon^-$, $f_\varepsilon = c_\varepsilon - A_\varepsilon^- d_\varepsilon$. Using $\lim_{\varepsilon\downarrow 0} \|A_\varepsilon^- - A_0^-\|$ from Theorem 8 and the assumptions on c_ε, d_ε we obtain

$$\lim_{\varepsilon\downarrow 0} \|f_\varepsilon - f_0\|_{d\alpha,\Gamma} = 0.$$

Due to Theorem 7 the operator $\mathcal{L}_\varepsilon$, $\varepsilon \geq 0$ has a bounded inverse. Moreover, Corollary 2 shows that

$$\lim_{\varepsilon\downarrow 0} \|\mathcal{L}_\varepsilon - \mathcal{L}_0\| = 0$$

in the operator norm induced by $T_d^{0\alpha}(\Gamma)$. Thus

$$\lim_{\varepsilon\downarrow 0} \|\mathcal{L}_\varepsilon^{-1} - \mathcal{L}_0^{-1}\| \leq \lim_{\varepsilon\downarrow 0} \left(\|\mathcal{L}_\varepsilon^{-1}\| \, \|\mathcal{L}_\varepsilon - \mathcal{L}_0\| \, \|\mathcal{L}_0^{-1}\|\right) = 0$$

and since $e_\varepsilon = \mathcal{L}_\varepsilon^{-1} f_\varepsilon$, $\varepsilon \geq 0$, we conclude

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \|e_\varepsilon - e_0\|_{d\alpha,\Gamma} &= \lim_{\varepsilon\downarrow 0} \|\mathcal{L}_\varepsilon^{-1} e_\varepsilon - \mathcal{L}_0^{-1} e_0\| \\
&\leq \lim_{\varepsilon\downarrow 0} \left(\|\mathcal{L}_\varepsilon^{-1} - \mathcal{L}_0^{-1}\| \, \|f_\varepsilon\|_{d\alpha,\Gamma} + \|\mathcal{L}_0^{-1}\| \, \|f_\varepsilon - f_0\|_{d\alpha,\Gamma}\right) \\
&= 0.
\end{aligned}$$

so

$$\lim_{\varepsilon\downarrow 0} \|n \wedge E_\varepsilon^+ - n \wedge E_0^+\|_{d\alpha,\Gamma} = 0.$$

Now we turn our attention to the magnetic fields, By the definition of the operators A_ε^+ we get,

$$n \wedge H_\varepsilon^+|_\Gamma - n \wedge H_0^+|_\Gamma = A_\varepsilon^+ e_\varepsilon - A_0^+ e_0.$$

Due to the continuity and the convergence properties of A_ε^+ we conclude

$$\lim_{\varepsilon\downarrow 0} \|n \wedge H_\varepsilon^+ - n \wedge H_0^+\|_{d\alpha,\Gamma} = 0.$$

The rest of the assertion is shown with the help of the boundary conditions on Γ

$$\begin{aligned}
n \wedge E_\varepsilon^+ - n \wedge E_0^+ &= n \wedge E_\varepsilon^- - n \wedge E_0^- + d_\varepsilon - d_0, \\
n \wedge H_\varepsilon^+ - n \wedge H_0^+ &= n \wedge H_\varepsilon^- - n \wedge H_0^- + c_\varepsilon - c_0.
\end{aligned}$$

■

Theorem 11 *Under the assumptions of Lemma 16 holds*

$$\lim_{\varepsilon\downarrow 0} \|H_\varepsilon^\pm - H_0^\pm\|_{0\alpha,\bar{D}^\pm} = 0, \qquad \lim_{\varepsilon\downarrow 0} \|E_\varepsilon^\pm - E_0^\pm\|_{0\alpha,\bar{D}^\pm} = 0.$$

Proof: E_ε^+ satisfies

$$\begin{aligned}
\triangle E_\varepsilon^+ + (k_\varepsilon^+)^2 E_\varepsilon^+ &= 0 \quad \text{in } D^+, \\
\operatorname{div} E_\varepsilon^+ &= 0 \quad \text{on } \Gamma, \\
\operatorname{curl} E_\varepsilon^+(x) \wedge \frac{x}{|x|} - i k_\varepsilon^+ E_\varepsilon^+(x) &= o\left(\frac{1}{|x|}\right), \quad |x| \to \infty
\end{aligned}$$

with $k_\varepsilon^+ = \sqrt{\omega^2\mu^+\varepsilon}$ and E_0^+ solves

$$\begin{aligned} \triangle E_0^+ &= 0 && \text{in } D^+, \\ \operatorname{div} E_0^+ &= 0 && \text{on } \Gamma, \\ E_0^+(x) &= o(1), && |x| \to \infty. \end{aligned}$$

Moreover

$$\int_{\Gamma_j} n(y) \cdot E_0^+(y) ds(y) = 0 \qquad \forall j \in \{1, \dots, m\}.$$

Using Theorem 5.9 from [1] and the convergence of the tangential fields provided by the previous lemma, we arrive at

$$\lim_{\varepsilon\downarrow 0} \|E_\varepsilon^+ - E_0^+\|_{0\alpha,\bar{D}^+} = 0, \qquad \lim_{\varepsilon\downarrow 0} \|\operatorname{curl} E_\varepsilon^+ - \operatorname{curl} E_0^+\|_{0\alpha,\bar{D}^+} = 0$$

and because of curl $E_\varepsilon^+ = i\omega\mu^+ H_\varepsilon^+$ it, is clear that $\lim_{\varepsilon\downarrow 0} \| H_\varepsilon^+ - H_0^+\|_{0\alpha,\bar{D}^+} = 0$.

In order to show convergence of the fields in the domain D^- we use the representations of the fields given in Lemma 12,

$$H_\varepsilon^- = -M_{k_\varepsilon^-}^-(n \wedge H_\varepsilon^-) - \frac{1}{i\omega\mu^-} T_{k_\varepsilon^-}^-(n \wedge E_\varepsilon^-), \qquad k_\varepsilon^- = \sqrt{\omega^2\mu^-\varepsilon + i\omega\sigma^-\mu^-}.$$

Thus

$$\begin{aligned} H_\varepsilon^- - H_0^- &= -M_{k_\varepsilon^-}^-(n \wedge H_\varepsilon^- - n \wedge H_0^-) - (M_{k_\varepsilon^-}^- - M_{k_0^-}^-)(n \wedge H_0^-) \\ &\quad - \frac{1}{i\omega\mu^-} T_{k_\varepsilon^-}^-(n \wedge E_\varepsilon^- - n \wedge E_0^+) - \frac{1}{i\omega\mu^-}(T_{k_\varepsilon^-}^- - T_{k_0^-}^-)(n \wedge E_0^-). \end{aligned}$$

Lemma 16 together with Lemma 32 and Lemma 33 from Appendix I3 provides $\lim_{\varepsilon\downarrow 0} \|H_\varepsilon^- - H_0^-\|_{0\alpha,\bar{D}^-} = 0$.

In completely the same way $\lim_{\varepsilon\downarrow 0} \|E_\varepsilon^- - E_0^-\|_{0\alpha,\bar{D}^-} = 0$ is shown. ∎

Corollary 3 *Let $H_\varepsilon^\pm, E_\varepsilon^\pm$ be the solution of Problem 1 to the boundary values $c_\varepsilon, d_\varepsilon \in T_d^{0\alpha}(\Gamma)$,*

$$\lim_{\varepsilon\downarrow 0} \|c_\varepsilon - c_0\|_{d\alpha,\Gamma} = 0, \qquad \lim_{\varepsilon\downarrow 0} \|d_\varepsilon - d_0\|_{d\alpha,\Gamma} = 0.$$

Then $H_\varepsilon^\pm$, $E_\varepsilon^\pm$ converge in $C^{0\alpha}(\bar{D}^\pm)$ to a solution $H_0^\pm$, $E_0^\pm$ of Problem 3 to the boundary values $c_0, g_0 = -\frac{1}{i\omega} Div\, d_0$ with circulations $h = (h_1, \dots, h_p)^T$ given by the linear system (9) of Theorem 5.

Proof: The above result follows directly from Theorem 11 and Theorem 5. ∎

Now we are coming back to Problem 4 resp. Problem 5, where we prescribe a current density J in the outer domain D^+ and homogeneous transmission conditions on Γ. In order to apply the previous convergence results to this kind of transmission problem we use volume potentials to transform the inhomogeneity J of the differential equations into inhomogeneities c_ε, d_ε of the boundary conditions.

Lemma 17 *Let J be given as in Problem 4 resp. Problem 5. For $\epsilon \geq 0$ there exist $H_\varepsilon^J, E_\varepsilon^J \in C^1(\mathbb{R}^3) \cap C^{0\alpha}(\mathbb{R}^3)$ such that*

$$curl\, H_\varepsilon^J = J - i\omega\varepsilon E_\varepsilon^J, \qquad curl\, E_\varepsilon^J = i\omega\mu^+ H_\varepsilon^J, \qquad div\, E_\varepsilon^J = 0 \qquad in\ D^+,$$

$$\int_{\Gamma_j} n \cdot E_\varepsilon^J ds = 0 \qquad \forall j \in \{1, \ldots, m\} \tag{37}$$

with

$$\omega\mu^+ H_\varepsilon^J(x) \wedge \frac{x}{|x|} - k_\varepsilon^+ E_\varepsilon^J(x) = O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty$$

for $\varepsilon > 0$ respectively

$$H_0^J(x) = o(1), \qquad E_0^J(x) = o(1), \qquad |x| \to \infty$$

for $\varepsilon = 0$. Moreover

$$\int_{\gamma_i^+} \tau \cdot H_0^J dl = 0 \qquad \forall i \in \{1, \ldots, p\} \tag{38}$$

and

$$\lim_{\varepsilon \downarrow 0} \|H_\varepsilon^J - H_0^J\|_{0\alpha, \bar{D}^+} = 0, \qquad \lim_{\varepsilon \downarrow 0} \|E_\varepsilon^J - E_0^J\|_{0\alpha, \bar{D}^+} = 0.$$

Proof: Throughout the whole proof we use k instead of k_ε^+. For $\varepsilon \geq 0$ we define

$$E_\varepsilon^J(x) = i\omega\mu^+ \int_{D^J} J(y)\Phi_k(x,y)dv(y), \tag{39}$$

$$H_\varepsilon^J(x) = \operatorname{curl}_x \int_{D^J} J(y)\Phi_k(x,y)dv(y). \tag{40}$$

Lemma 35 from Appendix B provides $H_\varepsilon^J, E_\varepsilon^J \in C^1(\mathbb{R}^3) \cap C^{0\alpha}(\mathbb{R}^3)$. and

$$\lim_{\varepsilon \downarrow 0} \|H_\varepsilon^J - H_0^J\|_{0\alpha, \bar{D}^+} = 0, \qquad \lim_{\varepsilon \downarrow 0} \|E_\varepsilon^J - E_0^J\|_{0\alpha, \bar{D}^+} = 0.$$

By definition curl $E_\varepsilon^J(x) = i\omega\mu^+ H_\varepsilon^J(x)$ and because of div $J = 0$, $J|_{\partial D^J} = 0$ we obtain using Lemma 28 from Appendix A

$$\left(\operatorname{div} E_\varepsilon^J\right)(x) = -i\omega\mu^+ \int_{\partial D^J} n(y) \cdot J(y)\Phi_k(x,y)ds(y) = 0,$$

i.e.

$$\begin{aligned}
\left(\operatorname{curl} H_\varepsilon^J\right)(x) &= (\operatorname{grad}_x \operatorname{div}_x - \Delta_x) \int_{D^J} J(y)\Phi_k(x,y)dv(y) \\
&= \frac{1}{i\omega\mu^+}\operatorname{grad}\operatorname{div} E_\varepsilon^J + J(x) + \frac{k^2}{i\omega\mu^+} E_\varepsilon^J(x) \\
&= J(x) - i\omega\varepsilon E_\varepsilon^J(x).
\end{aligned}$$

In the case $\varepsilon = 0$ the boundedness of D^J provides $H_0^J(x), E_0^J(x) = o(1)$ as $|x| \to \infty$. For $\varepsilon > 0$ it is shown in [7] that the fields $E_\varepsilon^J, H_\varepsilon^J$ meet the corresponding radiation condition.

An application of the Gaussian theorem yields

$$\int_{\Gamma_j} n(y) \cdot E^J_\varepsilon(y) ds(y) = \int_{D^-_j} \left(\operatorname{div} E^J_\varepsilon\right)(y) dv(y) = 0$$

for arbitrary $j \in \{1, \cdots, m\}$, where the D^-_j are the connected components of the interior domain D^-.

Moreover $\gamma_i^+ = \partial \Sigma_i^-$, $\Sigma_i^- \subset D^-$ and by Stokes' Theorem

$$\int_{\gamma_i^+} \tau \cdot H^J_0 dl = \int_{\Sigma_i^-} n \cdot \operatorname{curl} H^J_0 ds = \int_{\Sigma_i^-} n \cdot J ds = 0$$

since $\operatorname{supp}(J) \subset D^J \subset D^+$. ∎

By direct calculations the following result is easily verified.

Lemma 18 *Let $\tilde{H}^\pm$, $\tilde{E}^\pm$ be a solution of Problem 1 to $c = -n \wedge H^J_\varepsilon$, $d = -n \wedge E^J_\varepsilon$. Then*

$$H^+ = \tilde{H}^+ + H^J_\varepsilon, \qquad E^+ = \tilde{E}^+ + E^J_\varepsilon, \qquad H^- = \tilde{H}^-, \qquad E^- = \tilde{E}^-$$

are a solution of Problem 4.

Lemma 19 *Let $\tilde{H}^\pm$, $\tilde{E}^\pm$ be a solution of Problem 3 to $c = -n \wedge H^J_0$, $g = \frac{1}{i\omega} Div\,(n \wedge E^J_0)$ and circulations $h = (h_1, \ldots, h_p)^T \in \mathbb{C}^p$. Then*

$$H^+ = \tilde{H}^+ + H^J_0, \qquad E^+ = \tilde{E}^+ + E^J_0, \qquad H^- = \tilde{H}^-, \qquad E^- = \tilde{E}^-$$

solve Problem 5 with the same circulations h.

Proof: $H^\pm$, $E^\pm$ fulfill the differential equations and radiation conditions of Problem 5. Due (37),(38) we obtain

$$\begin{aligned} \int_{\Gamma_j} n \cdot E^+ ds &= 0 \qquad \forall j \in \{1, \ldots, m\}, \\ \int_{\gamma_i^+} \tau \cdot H^+ dl &= h_i \qquad \forall i \in \{1, \ldots, p\}, \end{aligned}$$

where h_i are the circulations of $\tilde{H}^+$. Moreover on Γ we have

$$n \wedge H^+ - n \wedge H^- = n \wedge \tilde{H}^+ - n \wedge \tilde{H}^- + n \wedge H^J_0 = c + n \wedge H^J_0 = 0$$

and since $g = \frac{1}{i\omega} \operatorname{Div}(n \wedge E^J_0) = -\frac{1}{i\omega} n \cdot \operatorname{curl} E^J_0 = -n \cdot (\mu^+ H^J_0)$ we deduce

$$n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) = n \cdot (\mu^+ \tilde{H}^+) - n \cdot (\mu^- \tilde{H}^-) + n \cdot (\mu^+ H^J_0) = g + n \cdot (\mu^+ H^J_0) = 0.$$

∎

As a direct consequence we obtain

Lemma *20 Problem 4 is uniquely solvable. Problem 5 is solvable and $H^{\pm}$, E^{-} are uniquely determined.*

Proof: Existence is obtained by using the existence results of Problem 1 resp. Problem 3 together with the last two Lemmata, The uniqueness results of these problems directly carry over to Problem 4 resp. Problem 5. ∎

Now we prove the final convergence theorem.

Theorem *12 The unique solution $H_\varepsilon^{\pm}$, $E_\varepsilon^{\pm}$ of Problem 4 converges in $C^{0\alpha}(\bar{D}^{\pm})$ to $H_0^{\pm}$, $E_0^{\pm}$ being a solution of Problem 5 with circulations h given by the nonsingular linear system $Ah = b$ with coefficients*

$$a_{il} = \int_\Gamma \left(n \wedge (E_l^+ - E_l^-)\right) \cdot Z_i^+ ds, \qquad b_i = -\int_\Gamma \left(n \wedge (E_\star^+ - E_\star^-)\right) \cdot Z_i^+ ds,$$

$i, l \in \{1, \ldots, p\}$, where $H_\star^{\pm}$, $E_\star^{\pm}$ resp. $H_l^{\pm}, E_l^{\pm}$, $l \in \{1, \ldots, p\}$ are arbitrary solutions of Problem 5 for $h_i = 0$ resp. for $J = 0$, $h_i = \delta_{li}$, $i \in \{1, \ldots, p\}$.

Proof: From Lemma 18 we know, that the unique solution $H_\varepsilon^{\pm}$, $E_\varepsilon^{\pm}$ of Problem 4 can be written as

$$H_\varepsilon^+ = \tilde{H}_\varepsilon^+ + H_\varepsilon^J, \qquad E_\varepsilon^+ = \tilde{E}_\varepsilon^+ + E_\varepsilon^J, \qquad H_\varepsilon^- = \tilde{H}_\varepsilon^-, \qquad E_\varepsilon^- = \tilde{E}_\varepsilon^-$$

where $\tilde{H}_\varepsilon^{\pm}$, $\tilde{E}_\varepsilon^{\pm}$ is the unique solution of Problem 1 for $c_\varepsilon = -n \wedge H_\varepsilon^J$, $d_\varepsilon = -n \wedge E_\varepsilon^J$. Due to Lemma 17 H_ε^J, E_ε^J converge in $C^{0\alpha}(\bar{D}^+)$ to H_0^J, E_0^J and thus c_ε, d_ε converge in $T_d^{0\alpha}(\Gamma)$ to $c_0 = -n \wedge H_0^J$, $d_0 = -n \wedge E_0^J$. According to Corollary 3, $\tilde{H}_\varepsilon^{\pm}$, $\tilde{E}_\varepsilon^{\pm}$ converge in $C^{0\alpha}(\bar{D}^{\pm})$ to $\tilde{H}_0^{\pm}$, $\tilde{E}_0^{\pm}$, which solve Problem 3 for the boundary data c_0, $g_0 = -\frac{1}{i\omega}\mathrm{Div}\, d_0$. Therefore $H_\varepsilon^{\pm}$, $E_\varepsilon^{\pm}$ converge in $C^{0\alpha}(\bar{D}^{\pm})$ to $H_0^{\pm}$, $E_0^{\pm}$,

$$H_0^+ = \tilde{H}_0^+ + H_0^J, \qquad E_0^+ = \tilde{E}_0^+ + E_0^J, \qquad H_0^- = \tilde{H}_0^-, \qquad E_0^- = \tilde{E}_0^-,$$

which is due to Lemma 19 a solution of Problem 5 with circulations $h_i = \tilde{h}_i = \int_{\gamma_i^+} \tau \cdot \tilde{H}_0^+ dl$.

Let $H_\star^{\pm}, E_\star^{\pm}$ resp. $H_l^{\pm}, E_l^{\pm}$, $l \in \{1, \ldots, p\}$ be given as in the assumption and define

$$\tilde{H}_\star^+ = H_\star^+ - H_0^J, \qquad \tilde{E}_\star^+ = E_\star^+ - E_0^J, \qquad \tilde{H}_\star^- = H_\star^-, \qquad \tilde{E}_\star^- = E_\star^-.$$

Then $\tilde{H}_\star^{\pm}, \tilde{E}_\star^{\pm}$ resp. $H_l^{\pm}, E_l^{\pm}$, $l \in \{1, \ldots, p\}$ are solutions of Problem 3 to the data c_0, g_0, $h_i = 0$ resp. to $c = 0$, $g = 0$, $h_i = \delta_{li}$, $i \in \{1, \ldots, p\}$. According to Corollary 3, the circulations $h = (h_1, \ldots, h_p)^T$ of H_0^+ are given by the nonsingular system $Ah = b$ with coefficients

$$a_{il} = \int_\Gamma \left(n \wedge (E_l^+ - E_l^-)\right) \cdot Z_i^+ ds, \qquad b_i = \int_\Gamma \left(d_0 - n \wedge (\tilde{E}_\star^+ - \tilde{E}_\star^-)\right) \cdot Z_i^+ ds,$$

$i, l \in \{1, \ldots, p\}$. Since $d_0 = -n \wedge E_0^J$ we finally get

$$b_i = -\int_\Gamma \left(n \wedge \left(E_0^J + \tilde{E}_\star^+ - \tilde{E}_\star^-\right)\right) \cdot Z_i^+ ds = -\int_\Gamma \left(n \wedge (E_\star^+ - E_\star^-)\right) \cdot Z_i^+ ds$$

which completes the proof. ∎

10 Appendix A

The following four Lemmata can be found in [I].

Lemma 21 *Let $a \in T_d^{0\alpha}(\Gamma)$. Then $\int_\Gamma Div\ a\ ds = 0$ and for $x \in \mathbb{R}^3 \setminus \Gamma$ holds*

$$\int_\Gamma div_x\,(a(y)\Phi_k(x,y))\,ds(y) = \int_\Gamma (Div\ a)(y)\Phi_k(x,y)ds(y).$$

Lemma 22 *Consider $D_{h_0}^{\pm} = \{x = z \pm hn(z) | z \in \Gamma, \quad 0 \le h \le h_0\}$ and E being C^1 in the interior of $D_{h_0}^+$ or $D_{h_0}^-$, so that curl E can be continuously extended to $D_{h_0}^+$ or $D_{h_0}^-$. Then*

$$Div\ (n \wedge E) = -n \cdot curl\ E\,|_\Gamma\ .$$

Lemma *23 For $Im(k) > 0$, $N(I \pm M_k) = \{0\}$. There exists a $\delta > 0$, so that for all $k \neq 0$, $Im(k) \geq 0$, $|k| < \delta$, $N(I \pm M_k) = \{0\}$. Moreover, for $k = 0$ we obtain*

$$\begin{aligned} N(I \pm M_0') &= span\left\{Z_1^{\pm}, \ldots, Z_p^{\pm}\right\}, \\ N(I \pm M_0) &= span\left\{n \wedge Z_1^{\mp}, \ldots, n \wedge Z_p^{\mp}\right\}. \end{aligned}$$

Lemma 24 *The matrices $Z^{\pm} = (z_{ij}^{\pm})$, $z_{ij}^{\pm} = \int_\Gamma Z_i^{\pm} \cdot \left(n \wedge Z_j^{\mp}\right) ds$, $i, j \in \{1, \ldots, p\}$, $Z_i^{\pm}$ being the Neumann jields of $D^{\pm}$, are nonsingular.*

Lemma *25 The operator M_0 maps $T_*^{0\alpha}(\Gamma)$ into itself and $I + M_0$ has a bounded inverse in $T_*^{0\alpha}(\Gamma)$.*

Proof: Let $a \in T_*^{0\alpha}(\Gamma)$and consider $F(x) = \int_\Gamma \text{curl}_x\ (a(y)\Phi_0(x,\ y))\ ds(y)$. For $x \in D+$

$$(\text{curl } F)(x) = \text{grad}_x \int_\Gamma (\text{Div } a)(y)\Phi_0(x,y)ds(y) = 0$$

since Div a = 0. By the jump conditions for the above layer potential we get $M_0 a = 2n \wedge F\,|_\Gamma - a$, so that

$$\text{Div } (M_0 a) = -2n \cdot \text{curl } F\,|_\Gamma = 0$$

by Lemma 22.

On the other hand, $N(I + M_0') = \text{span}\left\{Z_1^+, \ldots, Z_p^+\right\}$ and M_0, M_0' are adjoint with respect to (5). Therefore

$$\int_\Gamma M_0 a \cdot Z_i^+ ds = < M_0 a,\ Z_i^+ > = < a,\ M_0' Z_i^+ > = - < a,\ Z_i^+ > = - \int_\Gamma a \cdot Z_i^+ ds = 0$$

and the first part of the assertion is shown.

To show that $I + M_0$ has a bounded inverse in $T_*^{0\alpha}(\Gamma)$, we will use the Riesz theory. M_0 is compact in $T_d^{0\alpha}(\Gamma)$ and therefore in $T_*^{0\alpha}(\Gamma)$. Now assume, that a $\in T_*^{0\alpha}(\Gamma)$,

$(I + M_0)a = 0$. Using lemma 23, we know that a is given by $a = \sum_{l=1}^{p} x_l\, n \wedge Z_l^-$ and since $a \in T_*^{0\alpha}(\Gamma)$,

$$0 = \int_\Gamma a \cdot Z_i^+ ds = \sum_{l=1}^{p} x_l \int_\Gamma \left(n \wedge Z_l^-\right) \cdot Z_i^+ ds = \sum_{l=1}^{p} z_{il}^+ x_l.$$

But following Lemma 24, the $p \times p$ matrix with coefficients z_{il}^+ is nonsingular. Thus $a = 0$, $N(I + M_0) = \{0\}$ and $I + M_0$ has a bounded inverse in $T_*^{0\alpha}(\Gamma)$. ∎

Lemma 26 *The boundary value problem $F \in C^2(D^+) \cap C^{0\alpha}(\bar{D}^+)$*

$$\begin{aligned} \operatorname{curl} F = 0, \quad \operatorname{div} F &= 0 \qquad \text{in } D^+, \\ \int_{\Gamma_j} n \cdot F ds = 0 \quad & \quad \forall j \in \{1, \ldots, m\}, \\ n \wedge F = c & \qquad \text{on } \Gamma, \\ F(x) = o(1), & \qquad |x| \to \infty \end{aligned}$$

is uniquely solvable if and only if $c \in T_*^{0\alpha}(\Gamma)$. In this case there exists a constant c_α such that

$$\|F\|_{0\alpha, \bar{D}^+} \leq c_\alpha \|c\|_{d\alpha, \Gamma}.$$

Proof: Let F be a solution of the problem. By Lemma 22, we get $\operatorname{Div} c = 0$, so that $c \in T_d^{0\alpha}(\Gamma)$. Since $Z_i^+(x) = O\left(\frac{1}{|x|^2}\right)$ for $|x| \to \infty$ we can use Green's formula

$$\int_\Gamma c \cdot Z_i^+ ds = \int_\Gamma n \cdot \left(F \wedge Z_i^+\right) ds = -\int_{D^+} Z_i^+ \cdot \operatorname{curl} F - F \cdot \operatorname{curl} Z_i^+ dv = 0,$$

and c is also contained in $T_*^{0\alpha}(\Gamma)$.

Let us start the proof of the second direction by showing uniqueness. Assume F is a solution to $c = 0$. But then, F has to be a linear combination of Dirichlet fields $F = \sum_{j=1}^{m} f_j Y_j^+$. Since $Y_j^+ = \operatorname{grad} \varphi_j$, $\varphi_j|_{\Gamma_l} = \delta_{jl}$, we obtain

$$\int_{\Gamma_l} n \cdot Y_j^+ ds = \int_\Gamma n \cdot Y_j^+ \varphi_l ds = -\int_{D^+} \operatorname{div}\left(Y_j^+ \varphi_l\right) dv = -\int_{D^+} Y_j^+ \cdot Y_l^+ dv$$

and

$$\begin{aligned} 0 &= \sum_{l=1}^{m} \bar{f}_l \int_{\Gamma_l} n \cdot F ds \\ &= \sum_{j=1}^{m} \sum_{l=1}^{m} f_j \bar{f}_l \int_{\Gamma_l} n \cdot Y_j^+ ds \\ &= -\sum_{j=1}^{m} \sum_{l=1}^{m} f_j \bar{f}_l \int_{D^+} Y_j^+ Y_l^+ dv \\ &= -\int_{D^+} F \cdot \bar{F} dv, \end{aligned}$$

Thus F=O.

To show existence, we use the ansatz $F(x) = \int_\Gamma \text{curl}_x\ (a(y)\Phi_0(x,y))\,ds(y)$ for $a \in T_*^{0\alpha}(\Gamma)$. Due to the properties of the above layer potential [1], all conditions of the boundary value problem besides $n \wedge F = c$ on Γ are fulfilled. Taking the limit for $x \in D^+$ to Γ leads to $(I + M_0)a = 2c$, $a, c \in T_*^{0\alpha}(\Gamma)$ which is according to Lemma 25 uniquely solvable with $\|a\|_{d\alpha,\Gamma} \le c_\alpha \|c\|_{d\alpha,\Gamma}$. Since

$$\|F\|_{0\alpha\bar{D}^+} \le c_\alpha \|a\|_{d\alpha,\Gamma},$$

the proof is complete. ∎

Lemma 27 *For all $i \in \{1,\dots,p\}$ there exists a field $E \in C^1(D^+) \cap C^{0\alpha}(\bar{D}^+)$ with*

$$\text{curl } E = Z_i^+, \qquad \text{div } E = 0 \qquad \text{in } D^+,$$
$$\int_{\Gamma_j} n \cdot E\,ds = 0 \qquad \forall j \in \{1,\dots,m\},$$
$$E(x) = o(1) \qquad |x| \to \infty.$$

Proof: Let $e = n \wedge Z_i^+ \in T_d^{0\alpha}(\Gamma)$. Then $\text{Div } e = -n \cdot \text{curl } Z_i^+|_\Gamma = 0$ according to Lemma 22 and

$$\begin{aligned}
\int_\Gamma e \cdot Z_l^+ ds &= \int_\Gamma (n \wedge Z_i^+) \cdot Z_l^+ ds \\
&= -\int_{D^+} \text{div } (Z_i^+ \wedge Z_l^+) dv \\
&= -\int_{D^+} Z_l^+ \cdot \text{curl } Z_i^+ - Z_i^+ \cdot \text{curl } Z_l^+ dv \\
&= 0,
\end{aligned}$$

so that $e \in T_*^{0\alpha}(\Gamma)$. Thus we can solve the boundary value problem of Lemma 26 with $c = e$ and the solution F is of the form

$$F = \text{curl } \tilde{E}, \qquad \tilde{E}(x) = \int_\Gamma a(y)\Phi_0(x,y)ds(y)$$

with $a \in T_*^{0\alpha}(\Gamma)$. But for $H = F - Z_i^+$ holds

$$\text{curl } H = 0, \qquad \text{div } H = 0 \qquad \text{in } D^+,$$
$$\int_{\Gamma_j} n \cdot H\,ds = 0, \qquad \forall j \in \{1,\dots m\},$$
$$n \wedge H = 0 \qquad \text{on } \Gamma,$$
$$H(x) = o(1), \qquad |x| \to \infty,$$

so that H vanishes due to the uniqueness result of Lemma 26 and $F = Z_i^+$ resp. $\text{curl } \tilde{E} = Z_i^+$. In general $\int_{\Gamma_j} n \cdot \tilde{E}\,ds \ne 0$. But this is easily corrected by subtracting the gradient of the solution of the boundary value problem

$$\begin{aligned} \triangle u &= 0 && \text{in } D^+, \\ \partial_n u &= u_j && \text{on } \Gamma_j, \quad j \in \{1,\ldots,m\}, \\ u(x) &= o(1). && |x| \to \infty \end{aligned}$$

with $u_j = \int_{\Gamma_j} n \cdot \tilde{E} ds \left(\int_{\Gamma_j} ds \right)^{-1}$. This problem has a unique solution in $C^2(D^+) \cap C^{1\alpha}(\bar{D}^+)$ [1]. ■

Lemma 28 *Let $D \subset \mathbb{R}^3$ be open, bounded with ∂D being C^2. If $a \in C^1(D) \cap C(\bar{D})$ the volume potential*

$$V(x) = \int_D a(y)\, \Phi_k(x,y) dv(y)$$

lies in $C^1(\mathbb{R}^3)$ and

$$\begin{aligned} (curl\, V)(x) &= \int_D (curl\, a)(y)\, \Phi_k(x,y) dv(y) - \int_{\partial D} n(y) \wedge a(y)\, \Phi_k(x,y) ds(y), \\ (div\, V)(x) &= \int_D (div\, a)(y)\, \Phi_k(x,y) dv(y) - \int_{\partial D} n(y) \cdot a(y)\, \Phi_k(x,y) ds(y). \end{aligned}$$

If $a \in C^1(D) \cap C^{0\alpha}(\bar{D})$ then $V \in C^2(D)$.

This lemma, which is for example found in [6], is used to show the following useful result.

Lemma 29 *To every $\lambda \in C_*^{0\alpha}(\Gamma)$ exists an $a \in T_d^{0\alpha}(\Gamma)$ with*

$$Div\, a = \lambda, \qquad \|a\|_{d\alpha,\Gamma} \leq c_\alpha \|\lambda\|_{0\alpha,\Gamma}.$$

Proof: We first consider the boundary value problem

$$\triangle u = 0 \quad \text{in } D^-, \qquad \partial_n u = \lambda \quad \text{on } \Gamma. \tag{41}$$

Let us try to find a solution of (41) in form of a single layer potential

$$u(x) = \int_\Gamma \varphi(y) \Phi_0(x,y) ds(y) \tag{42}$$

with density $\varphi \in C_*^{0\alpha}(\Gamma)$. From the jump relations for (42) we obtain the integral equation

$$(I + K_0')\varphi = 2\lambda. \tag{43}$$

From Lemma 5 we know, that K_0' is compact in $C^{0\alpha}(\Gamma)$. Moreover, in the proof of Theorem 1 we have shown, that K_0' maps $C_*^{0\alpha}(\Gamma)$ into itself and thus is also compact in $C_*^{0\alpha}(\Gamma)$. According to [1], there exists a basis ψ_l of $N(I + K_0')$ with

$$\int_{\Gamma_j} \psi_l\, ds = \delta_{jl}, \qquad j, l \in \{1,\ldots,m\},$$

so that $N(I + K_0') = \{0\}$ in $C_*^{0\alpha}(\Gamma)$. Thus $I + K_0'$ has a bounded inverse in $C_*^{0\alpha}(\Gamma)$ and (43) has a unique solution $\varphi \in C_*^{0\alpha}(\Gamma)$ with

$$||\varphi||_{0\alpha,\Gamma} \leq c_1||\lambda||_{0\alpha,\Gamma}.$$

Using the properties of the single layer potential, we obtain

$$u, \text{grad } u \in C^{0\alpha}(\bar{D}^-), \qquad \max\{||u||_{0\alpha,\bar{D}^-}, ||\text{grad } u||_{0\alpha,\bar{D}^-}\} \leq c_2||\lambda||_{0\alpha,\Gamma}.$$

Now we consider the corresponding exterior boundary value problem

$$\triangle v = 0 \text{ in } D^+, \qquad \partial_n v = \lambda \text{ on } \Gamma, \qquad v(x) = o(1) \quad |x| \to \infty.$$

Using again a single layer potential ansatz, it is shown in [1], that a unique strong solution exists with

$$v, \text{grad v} \in C^{0\alpha}(\bar{D}^+), \qquad \max\{||v||_{0\alpha,\bar{D}^+}, ||\text{grad } v||_{0\alpha,\bar{D}^+}\} \leq c_3||\lambda||_{0\alpha,\Gamma}.$$

Let w = grad u, where u is the solution of (41). Then $w \in C^1(D^-) \cap C^{0\alpha}(\bar{D}^-)$ and curl w = 0, div w = 0 in D^-. For the volume potential

$$F(x) = \int_{D^-} w(y)\Phi_0(x, y)ds(y)$$

Lemma 28 now provides

$$\begin{aligned}
(\text{curl } F)(x) &= -\int_\Gamma n(y) \wedge w(y)\Phi_0(x,y)ds(y), \\
(\text{div } F)(x) &= -\int_\Gamma n(y) \cdot w(y)\Phi_0(x,y)ds(y).
\end{aligned}$$

Since w $|_\Gamma \in C^{0\alpha}(\Gamma)$ we get [1]

$$\text{curl } F, \text{div } F \in C^1(D^-) \cap C^{0\alpha}(\bar{D}^-)$$

with

$$||\text{curl } F||_{0\alpha,\bar{D}^-} \leq c_4||w||_{0\alpha,\bar{D}^-}, \qquad ||\text{div } F||_{0\alpha,\bar{D}^-} \leq c_5||w||_{0\alpha,\bar{D}^-}.$$

Using the continuous dependence of w on λ, we conclude

$$||\text{curl } F||_{0\alpha,\bar{D}^-} \leq c_6||\lambda||_{0\alpha,\Gamma}, \qquad ||\text{div } F||_{0\alpha,\bar{D}^-} \leq c_7||\lambda||_{0\alpha,\Gamma}.$$

Consider now the surface potential

$$G(x) = \int_\Gamma n(y)v(y)\Phi_0(x, y)ds(y).$$

Then $G \in C^2(D^-) \cap C^{0\alpha}(\bar{D}^-)$, curl G, div $G \in C^1(D^-) \cap C^{0\alpha}(\bar{D}^-)$ and

$$||\text{curl } G||_{0\alpha,\bar{D}^-} \leq c_8||v||_{0\alpha,\Gamma} \leq c_9||\lambda||_{0\alpha,\Gamma}.$$

In the final step, we consider $A = G - F \in C^2(\bar{D}^-)$. Then

$$\begin{aligned}(\operatorname{div} A)(x) &= (\operatorname{div} G)(x) - (\operatorname{div} F)(x)\\ &= \int_\Gamma v(y)\, n(y) \cdot \operatorname{grad}_x \Phi_0(x,y) ds(y) + \int_\Gamma n(y) \cdot w(y)\, \Phi_0(x,y) ds(y)\\ &= \int_\Gamma (\partial_n v)(y)\, \Phi_0(x,y) - v(y) \partial_{n(y)} \Phi_0(x,y) ds(y)\\ &= 0\end{aligned}$$

in D^- and for $B = \operatorname{curl} A = \operatorname{curl} G - \operatorname{curl} F$ holds

$$\|B\|_{0\alpha,\bar{D}^-} \le c_{10} \|\lambda\|_{0\alpha,\Gamma}$$

and

$$\operatorname{curl} B = -w \quad \text{in } \bar{D}^-.$$

Taking $a = n \wedge B|_\Gamma$ we see, that $a \in T_d^{0\alpha}(\Gamma)$ with

$$\operatorname{Div} a = -n \cdot \operatorname{curl} B = n \cdot w = n \cdot \operatorname{grad} u = \lambda$$

and

$$\|a\|_{d\alpha,\Gamma} \le c_\alpha \|\lambda\|_{0\alpha,\Gamma}.$$

■

11 Appendix B

In this section we consider some asymptotic properties of the single- and double layer potentials used in the convergence theorems. The first lemma is a slight modification of a result from [1].

Lemma 30 *Let $K(x,y)$ be continuous for all $x \in \mathbb{R}^3$, $y \in \Gamma$ with $x \neq y$. Assume*

$$|K(x,y)| \le C_1 |x-y|^{-1} + C_2$$

and

$$|K(x_1,y) - K(x_2,y)| \le C_3|x_1 - x_2| + C_4 \frac{|x_1 - x_2|}{|x_1 - y|} + C_5 \frac{|x_1 - x_2|}{|x_1 - y|^2} \tag{44}$$

for all $x, x_1, x_2 \in \mathbb{R}^3, y \in \Gamma$ such that $2|x_1 - x_2| \le |x_1 - y|$, where $C_1, \ldots, C_5$ are independent of x_1, x_2, y. Then the generalized potential

$$u(x) = \int_\Gamma K(x,y) c(y) ds(y)$$

with $c \in C(\Gamma)$ is well defined for all $x \in \mathbb{R}^3$. Moreover $u \in C^{0,\beta}(\mathbb{R}^3)$ for all $\beta \in (0,1)$ and

$$\|u\|_{0\beta} \le C_\beta \|c\|_{\infty,\Gamma}, \qquad C_\beta \le M \sum_{i=1}^{5} C_i, \tag{45}$$

where M only depends on Γ and β.

Proof: The first estimate ensures existence of the integral $u(x) = \int_\Gamma K(x, y)c(y)ds(y)$ with a weakly singular kernel. Using (44) the Hölder continuity is shown in complete analogy with [1]. ■

Lemma 31 *For* $c \in T_d^{0\alpha}(\Gamma)$, $x \in D^\pm$ *and* $Im\ (k) \geq 0$ *define*

$$\left(S_k^\pm c\right)(x) = \int_\Gamma c(y)\Phi_k(x, y)ds(y).$$

Then $S_k^\pm$ *can be extended to* $\bar{D}^\pm$ in a *Hölder* continuous *fashion such that*

$$\|S_k^\pm c\|_{0\alpha,\bar{D}^\pm} \leq c_\alpha \|c\|_{d\alpha,\Gamma}$$

with c_α being independent of c $\in T_d^{0\alpha}(\Gamma)$. Moreover

$$\lim_{k\to k_0} \|S_k^\pm - S_{k_0}^\pm\| = 0$$

holds in the induced operator norm.

Proof: Existence and extension properties are shown in [1]. For $x, y \in \mathbb{R}^3, x \neq y$ and $r = |x - y|$ define

$$K(x, y) = \frac{e^{ikr} - e^{ik_0 r}}{4\pi r}. \tag{46}$$

Using a meanvalue argument one immediately arrives at

$$|e^{ikr_1} - e^{ikr_2}| \leq 3|k|\,|r_1 - r_2| \tag{47}$$

for arbitrary $r_1, r_2 \geq 0$, Im $(k) \geq 0$, i.e.

$$|K(x,y)| \leq M_1|k - k_0|. \tag{48}$$

M_1 is independent of k, k_0, x and y. Considering the decomposition

$$\begin{aligned} K(x_1, y) - K(x_2,y) = {} & \frac{e^{ik_0 r_1} - e^{ik_0 r_2}}{4\pi r_1}(e^{i(k-k_0)r_1} - 1) \\ & + \frac{e^{ik_0 r_2}}{4\pi r_1}(e^{i(k-k_0)r_1} - e^{i(k-k_0)r_2}) \\ & + \frac{e^{ik_0 r_2}}{4\pi}(e^{i(k-k_0)r_2} - 1)\left(\frac{r_2 - r_1}{r_1 r_2}\right), \end{aligned}$$

$r_1 = |x_1 - y|$, $r_2 = |x_2 - y|$ and applying (47) we get

$$|K(x_1,y) - K(x_2, y)| \leq |k - k_0|\,|r_2 - r_1|(M_2\,|k_0| + \frac{M_3}{r_1}), \tag{49}$$

where M_2, M_3 are independent of k, k_0, x_1, x_2, y. By Lemma 30 and the special choice

$$\begin{array}{lll} C_1 = 0, & C_2 = M_1|k-k_0|, & C_3 = M_2|k-k_0|\,|k_0|, \\ C_4 = M_3|k-k_0|, & C_5 = 0, & \end{array}$$

we obtain for arbitrary $c \in T_d^{0\alpha}(\Gamma)$ $\|S_k^+ c - S_{k_0}^+ c\|_{0\alpha,\bar{D}^+} \leq C_\alpha \|c\|_{d\alpha,\Gamma}$ and $|k-k_0|$ enters each C_i, $i \in \{1,\ldots,5\}$ in a linear fashion. Therefore we finally conclude $\lim_{k\to k_0} \|S_k^+ - S_{k_0}^+\| = 0$ in the induced norm. The corresponding result for $S_k^-, S_{k_0}^-$ is shown in a similar manner. ∎

Lemma 32 *For $M_k^\pm$ from Lemma 11 holds*

$$\lim_{k\to k_0} \|M_k^\pm - M_{k_0}^\pm\| = 0$$

in the corresponding induced operator norm.

Proof: Because of

$$\operatorname{curl}_x (c(y)\Phi_k(x,y) - c(y)\Phi_{k_0}(x,y)) = \operatorname{grad}_x (\Phi_k(x,y) - \Phi_{k_0}(x,y)) \wedge c(y)$$

we consider the integral kernel

$$G(x,y) = \frac{x-y}{r} ik_0 K(x,y) + \frac{x-y}{4\pi r^2} i(k-k_0)e^{ikr} - \frac{x-y}{r^2} K(x,y),$$

where $K(x,y)$ was introduced in (46). This leads to

$$|G(x,y)| \leq M_2|k-k_0|\,|k_0| + M_1|k-k_0|\,|x-y|^{-1}$$

for all $x, y \in \mathbb{R}^3$, $x \neq y$. M_1, M_2 are independent of k, k_0, x und y. Given $x_1, x_2, y \in \mathbb{R}^3$ and $r_1 = |x_1 - y| > 0$, $r_2 = |x_2 - y| > 0$, we can decompose

$$\begin{aligned} G(x_1,y) - G(x_2,y) = & \\ = \quad & \left(\frac{x_1-y}{r_1} - \frac{x_2-y}{r_2}\right) ik_0 K(x_1,y) \\ + \quad & ik_0 \frac{x_2-y}{r_2}(K(x_1,y) - K(x_2,y))\, i(k-k_0)\left(\frac{x_1-y}{4\pi r_1^2} - \frac{x_2-y}{4\pi r_2^2}\right) e^{ikr_1} \\ + \quad & i(k-k_0)\frac{x_2-y}{4\pi r_2^2}(e^{ikr_1} - e^{ikr_2}) \\ - \quad & \left(\frac{x_1-y}{r_1^2} - \frac{x_2-y}{r_2^2}\right) K(x_1,y) \\ - \quad & \frac{x_2-y}{r_2^2}(K(x_1,y) - K(x_2,y)). \end{aligned}$$

Using (47) and (49) we see that

$$|K(x_1,y) - K(x_2,y)| \leq \left(M_4|k_0| + \frac{M_5}{r_1}\right)|k-k_0|\,|x_1-x_2|,$$

i.e.

$$|G(x_1,y)-G(x_2,y)| \leq \left(D_1|k_0|^2+D_2\frac{|k_0|}{r_1}+D_3\frac{|k|+|k_0|}{r_2}\right)|k-k_0|\,|x_1-x_2| + \left(\frac{D_4}{r_1r_2}+\frac{D_5}{r_1^2}\right)|k-k_0|\,|x_1-x_2|,$$

with some $D_1,\dots,D_5$ independent of k, k_0,x_1,x_2 and y. Given $x_1,x_2 \in \mathbb{R}^3$, $y \in \Gamma$, $r_1 = |x_1 - y| > 0$, $r_2 = |x_2 - y| > 0$ and $2|x_1 - x_2| \leq |x_1 - y|$, we obtain by applying Lemma 30 with the special choice

$$\begin{array}{ll} C_1 = M_1|k-k_0|, & C_2 = M_2|k-k_0|\,|k_0|, \\ C_3 = D_1|k-k_0|\,|k_0|^2, & C_4 = (D_2|k_0|+2D_3(|k|+|k_0|))\,|k-k_0|, \\ C_5 = (2D_4+D_5)|k-k_0| & \end{array}$$

the estimate

$$\|M_k^+c - M_{k_0}^+c\|_{0\alpha,\bar{D}^+} \leq C_\alpha\|c\|_{d\alpha,\Gamma}.$$

$k-k_0$ enters each C_i, $i \in \{1,\dots,5\}$ in a linear fashion. Therefore (45) provides $\lim_{k\to k_0}\|M_k^+ - M_{k_0}^+\| = 0$ in the induced operator norm. The convergence $\lim_{k\to k_0}\|M_k^- - M_{k_0}^-\| = 0$ is shown in a similar manner. ■

Lemma *33 For $T_k^\pm$ from Lemma 11 holds*

$$\lim_{k\to k_0}\|T_k^\pm - T_{k_0}^\pm\| = 0$$

in the corresponding induced operator norm.

Proof: For $c \in T_d^{0\alpha}(\Gamma)$ and $x \in D^\pm$ define

$$\left(F_k^\pm c\right)(x) = \int_\Gamma (\mathrm{Div}\ c)(y)\ \mathrm{grad}\ _x\Phi_k(x,y)ds(y).$$

Because of $c \in T_d^{0\alpha}(\Gamma)$, i.e. Div $c \in C^{0\alpha}(\Gamma)$, $F_k^\pm$ can be extended in a Hölder continuous fashion up to $\bar{D}^\pm$. Using the properties of $G(x,y) = \mathrm{grad}\ _x(\Phi_k(x,y) - \Phi_{k_0}(x,y))$ mentioned in the proof of Theorem **32** we obtain $\lim_{k\to k_0}\|F_k^\pm - F_{k_0}^\pm\| = 0$. Applying Lemma 31 we see that $\lim_{k\to k_0}\|k^2S_k^\pm - {k_0}^2S_{k_0}^\pm\| = 0$, i.e. $\lim_{k\to k_0}\|T_k^\pm - T_{k_0}^\pm\| = 0$. ■

Now we are in the position to consider the integral operators for the potentials on the boundary Γ.

Lemma *34 For the operators M_k, $T_k : T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$ from Definition 2 holds*

$$\lim_{k\to k_0}\|M_k - M_{k_0}\| = 0, \qquad \lim_{k\to k_0}\|T_k - T_{k_0}\| = 0$$

in the induced operator norm.

Proof: Lemma 32 and Lemma 33 provide

$$\|M_k c - M_{k_0} c\|_{0\alpha,\Gamma} \le d_1(k)\|c\|_{d\alpha,\Gamma} \quad \|T_k c - T_{k_0} c\|_{0\alpha,\Gamma} \le d_2(k)\|c\|_{d\alpha,\Gamma},$$

where $d_1(k)$, $d_2(k)$ are independent of $c \in T_d^{0\alpha}(\Gamma)$ and $d_1(k), d_2(k) = o(1)$ as $k \to k_0$.

For $x \in D^-$ define

$$\begin{aligned} F(x) &= \operatorname{curl}_x \int_\Gamma \operatorname{curl}_x (c(y)\Phi_k(x,y))\, ds(y) \\ &= \operatorname{grad}_x \int_\Gamma (\operatorname{Div} c)(y)\Phi_k(x,y)ds(y) + k^2 \int_\Gamma c(y)\Phi_k(x,y)ds(y). \end{aligned}$$

In connection with Lemma 22 from Appendix A we see

$$\begin{aligned} (\operatorname{Div} M_k c)(x) &= -n(x)\cdot \int_\Gamma (\operatorname{Div} c)(y)\operatorname{grad}_x \Phi_k(x,y)ds(y) - (\operatorname{Div} a)(x) \\ &\quad -k^2 n(x)\cdot \int_\Gamma c(y)\Phi_k(x,y)ds(y). \end{aligned}$$

Therefore Lemma 33 leads to $\|\operatorname{Div} M_k c - \operatorname{Div} M_{k_0} c\|_{0\alpha,\Gamma} \le d_3(k)\|c\|_{d\alpha,\Gamma}$, where $d_3(k)$ is independent of $c \in T_d^{0\alpha}(\Gamma)$ and $d_3(k) = o(1)$ as $k \to k_0$. Thus $\lim_{k\to k_0} \|M_k - M_{k_0}\| = 0$ in the induced operator norm in $T_d^{0\alpha}(\Gamma)$.

In order to show convergence for T_k we define for $x \in D^-$

$$\begin{aligned} G(x) &= \operatorname{curl}_x \int_\Gamma (\operatorname{Div} c)(y)\operatorname{grad}_x \Phi_k(x,y)ds(y) + k^2 \operatorname{curl}_x \int_\Gamma c(y)\Phi_k(x,y)ds(y) \\ &= k^2 \operatorname{curl}_x \int_\Gamma c(y)\Phi_k(x,y)ds(y). \end{aligned}$$

Using again Lemma 22 from Appendix A we conclude for $x \in \Gamma$

$$(\operatorname{Div} T_k c)(x) = -n(x)\cdot G(x) = -k^2 n(x)\cdot \int_\Gamma \operatorname{curl}_x (c(y)\Phi_k(x,y))\, ds(y).$$

Lemma 32 states that $\|\operatorname{Div} T_k c - T_{k_0} c\|_{0\alpha,\Gamma} \le d_4(k)\|c\|_{d\alpha,\Gamma}$ with $d_4(k)$ being independent of $c \in T_d^{0\alpha}(\Gamma)$, $d_4(k) = o(1)$ as $k \to k_0$. Finally we arrive at $\lim_{k\to k_0} \|T_k - T_{k_0}\| = 0$ in the induced operator norm in $T_d^{0\alpha}(\Gamma)$. ∎

Lemma 35 *Let $J \in C^1(\mathbb{R}^3)$ and D^J open, bounded, $\bar{D}^J \subset D^+$, ∂D^J being C^2, such that supp$(J) \subset D^J$. The volume potential*

$$V_k(x) = \int_{D^J} J(y)\Phi_k(x,y)dv(y)$$

satisfies V_k, curl $V_k \in C^1(\mathbb{R}^3) \cap C^{0\alpha}(\mathbb{R}^3)$ and

$$\lim_{k\to 0} \|V_k - V_0\|_{0\alpha,\bar{D}^+} = 0, \qquad \lim_{k\to 0} \|\mathit{curl}\, V_k - \mathit{curl}\, V_0\|_{0\alpha,\bar{D}^+} = 0.$$

Proof: Due to the assumptions we have $J \in C^1(\mathbb{R}^3) \cap C^{0\alpha}(\mathbb{R}^3)$ and the regularity of V_k can be proven using Theorem 8.1 from [2].

Analysing the convergence behaviour we can write

$$V_k(x) - V_0(x) = \int_{D^J} J(y)(\Phi_k(x,y) - \Phi_0(x,y))dv(y) = \int_{D^J} J(y)K(x,y)dv(y),$$

where $K(x,y)$ is given by (46) and fulfills due to (48), (49) for $x, x_1, x_2 \in \mathbb{R}^3$, $y \in D^J$

$$|K(x,y)| \leq M_1|k|, \tag{50}$$

$$|K(x_2,y) - K(x_1,y)| \leq M_2|k|\frac{|x_1 - x_2|}{|x_1 - y|}, \tag{51}$$

where M_1 is independent of k, x, y and M_2 is independent of x_1, x_2, y, k. From (50) we immediately get

$$\|V_k - V_0\|_{\infty,\bar{D}+} \leq M_3|k|,$$

M_3 being independent of k. Considering $B = \{x \in \mathbb{R}^3 \mid |x_1 - y| < 1\}$ we conclude

$$\begin{aligned}\int_{D^J} \frac{1}{|x_1 - y|} dv(y) &\leq \int_B \frac{1}{|x_1 - y|} dv(y) + \int_{D^J \setminus B} \frac{1}{|x_1 - y|} dv(y)\\ &\leq 2\pi + \mathrm{vol}(D^J),\end{aligned}$$

and by (52)

$$\begin{aligned}|(V_k(x_2) - V_0(x_2)) - (V_k(x_1) - V_0(x_1))| &\leq \|J\|_\infty \int_{D^J} |K(x_2,y) - K(x_1,y)|dv(y)\\ &\leq M_4|k|\,|x_1 - x_2|\end{aligned}$$

where M_4 is independent of x_1, x_2 and k. Therefore we have

$$\lim_{k \to 0} \|V_k - V_0\|_{0\alpha,\bar{D}+} = 0.$$

By Lemma 28 from Appendix A we know that

$$(\mathrm{curl}\ V_k)(x) = \int_{D^J} (\mathrm{curl}\ J)(y)\Phi_k(x,y)(x,y)dv(y).$$

Applying the same arguments as above provides

$$\lim_{k \to 0} \|\mathrm{curl}\ V_k - \mathrm{curl}\ V_0\|_{0\alpha,\bar{D}+} = 0.$$

■

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